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Florentin Smarandache

Mohamed Abdel-Basset

Said Broumi

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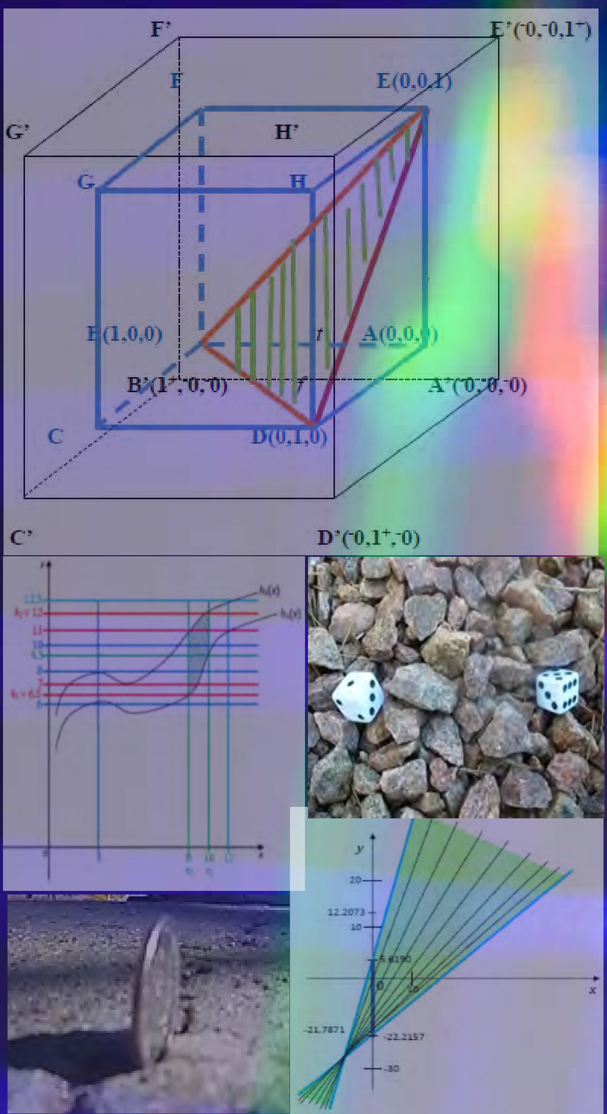

<A> <neutA> <antiA>

Florentin Smarandache . Mohamed Abdel-Basset . Said Broumi
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The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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March 20, 2019

Prof. Florentin Smarandache
Univ New Mexico, Gallup Campus

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NeutroAlgebra for the evaluation of barriers to migrants' access in Primary Health Care in Chile based on PROSPECTOR function

Diego Silva Jiménez[1], Juan Alexis Valenzuela Mayorga[2], and Mara Esther Roja Ubilla[3], Noel Batista Hernández[4]

[1] Centro de Salud Pública, Universidad Central de Chile, Santiago, Chile. E-mail: email: diego.silva@ucentral.cl
[2] Centro de Salud Pública, Universidad Central de Chile, Santiago, Chile. E-mail: javalenzuelam@ucentral.cl

[3] Servicio de Salud Metropolitano Sur Oriente, Santiago, Chile. E-mail: marojas@ssmso.cl

[4] Universidad de Guayaquil, Guayaquil, Ecuador. E-mail: noel.batistah@ug.edu.ec

Abstract. This research is the result of the CIP 2018016 project of the Universidad Central of Chile. The migrant population has beliefs, values and health practices different from Chilean people, for which Chilean citizens are not prepared as a society or health sector. The purpose of this research is to investigate which are the access barriers to health that the international migrant population faces in primary health care in Chile. To accomplish this objective, a group of migrants was surveyed on an evaluation scale between -10 to 10 in various aspects of access to health. A generalization of the well-known PROSPECTOR function was used as aggregator, where the aggregation between the two extreme values is undefined. Except for indefiniteness, this function is a truth membership of an offuninorm or a uninorm in the interval [-1, 1]. We preferred to keep the indeterminacy to take into account totally contradictory opinions. This turns the generalization of the PROSPECTOR function into a NeutroFunction, and this problem into an application of NeutroAlgebra.

Keywords: Access barriers, access to health, international migrant, primary care, PROSPECTOR, uninorm, offuninorm, NeutroFunction, NeutroAlgebr

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1. Introduction

The World Health Organization (WHO) states that a health system brings together all the institutions and organizations whose primary objective is to maintain and improve the health of the population. Most health systems are made up of different sectors, public, private, traditional and informal, and must provide good treatments and services that respond to the needs of the population and are fair from a financial point of view, [1].

The Chilean health system is mixed, both in insurance and in the provision of services. This is done through two subsystems, one private and one public. The private sector delivers closed and outpatient health actions of varying complexity, for profit. In the public sector, the provision of services is carried out through the National Health System, which has 29 services distributed throughout the country and must be registered in "Fondo Nacional de Salud" or National Fund of Health (FONASA in Spanish). Health care is provided at different levels of complexity, from primary to tertiary level, the first one is the point of access to the public sector through the Family Health Centers (CESFAM in Spanish), [2].

"Access to health services is the ability to get care when it is needed, [2, 3]. This can be determined by various factors and variables such as the location of health centers and the availability of medical or health providers (geographical or physical barriers), up to health insurance and health care costs, also can be influenced by cultural barriers or language.", [2].

The phenomenon of international migration to Chile had a slow beginning at the dawn of the 90s, from the neighboring countries, as the decades passed, it took on an unusual force, with international migrants in Chile today reaching 2.7% of the total population, [4]. The migrant population of the South East Metropolitan Health Service (SSMSO in Spanish) corresponds to 1.35%. The commune of Florida represents 39% of that population, which is the largest in the SSMSO and is the one with the highest vulnerability index, [5].

The arrival of migrants to the country may represent an opportunity to improve their health because "Chile has greater control of infectious and nutritional diseases compared to other Latin American countries, which can increase life expectancy" ([6]) as long as they can access the health system.

"The access difficulties of undocumented immigrants are reinforced by two aspects: the breach of ministerial agreements and the personal criteria of the agents", [7]. This means that the migrant population that is not assigned to any health service corresponds to 15.7%, while children under 14 years of age reach 26.6 of this percent, [2]. This means that immigrants are about 7.5 times more likely to have no health insurance than Chileans, which means that immigrants show less need for health than the natives do, which is accompanied at the time of requesting care. They do not attend any assistance and if they attend their health, then their demand is not satisfied ([8]). Low-income people have a lesser chance of accessing health services when they need it ([3]). This is the same as that of the indigenous population in Guatemala ([9]), generating inequality and intersector inequity.

This research aims to evaluating the access barriers to health that the international migrant population faces in primary health care in Chile. To achieve this objective, a group of 28 international migrants of different sexes and nationalities who are cared at the Santa Amalia CESFAM of Florida Commune are surveyed, who belong to the Eastern Metropolitan Health Service. Respondents evaluated different relevant aspects in health care on a numerical scale with a maximum of 10 for approval and a minimum of -10 for disapproval. A generalization of the well-known PROSPECTOR function of the MYCIN medical expert system, is used, [10-12]. This function has proven effectiveness in practice as an aggregator. On the other hand, it is undefined when applied to the two extreme values -1 and 1, where a total contradiction is shown in the evaluation.

This function is a mapping from $[-1,1]^2$ into $[-1,1]$ that is classified as a uninorm in this interval, [13]. A uninorm is an aggregation operator that generalizes the notion of t-norm and t-conorm, where the axioms of commutativity, associativity, monotonicity, and the existence of a neutral element are maintained, [12-14]. Regarding this last axiom, the uninorm contains a neutral element different from 0 and 1. The PROSPECTOR has as a neutral element 0 if it extends to the interval $[-1,1]$ and 0.5 if it is rescaled to the interval $[0,1]$. Except for the indefiniteness, this function is considered the truth membership function of an offuninorm, [15], which are uninorms defined for neutrosophic offsets, [16-19], which are neutrosophic sets with truth values >1 or <0. This theory generalizes the notion of neutrosophic uninorms, [20, 21], which in turn generalize the notion of fuzzy uninorm and of uninorms defined in intuitionistic fuzzy sets, [22-25].

In this paper we prefer to keep undefined the aggregation of the values -1 and 1, because we want to consider any uncertainty or indeterminacy in the evaluation of health care in Chile. This feature of the generalized PROSPECTOR function turns it into a NeutroFunction, and therefore the operations between the elements become operations within a NeutroAlgebra, since it does not satisfy the associativity axiom. A NeutroAlgebra is an algebra which has at least one NeutroOperation or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for the other elements), [26-28]. Some neutrosophic approaches to algebra can be read in [29-35].

This paper is divided into the following sections. Section 2 is dedicated to recalling the main concepts of NeutroAlgebra and PROSPECTOR function. Section 3 presents the method that we will use to measure the situation of access to health according to the surveyed migrants, as well as the results obtained. This ends with the conclusions. Appendix A contains a table of a NeutroAlgebra generated by generalizing the PROSPECTOR function[36].

2. Preliminaries

This section contains the main concepts of NeutroAlgebra and PROSPECTOR function.

The algebraic structures were extended by Smarandache (in 2019 and 2020) to NeutroAlgebras and AntiAlgebras [24-26].

Definition 1) [26]: Let X be a given nonempty space (or simply set) included into a universe of discourse U. Let <A> be an item (concept, attribute, idea, proposition, theory, etc.) defined on the set X. Through the process of neutrosophication, we split the set X into three regions [two opposite ones <A> and <antiA>, and one neutral (indeterminate) <neutroA> between them], regions which may or may not be disjoint – depending on the application, but they are exhaustive (their union equals the whole space
A *NeutroAlgebra* is an algebra which has at least one *NeutroOperation* or one *NeutroAxiom* (axiom that is true for some elements, indeterminate for other elements, and false for other elements).

The NeutroAlgebra is a generalization of *Partial Algebra*, which is an algebra that has at least one *Partial Operation*, while all its Axioms are totally true (classical axioms).

Definition 2) [26].: A function f: X → Y is called a *Partial Function* if it is well-defined for some elements in X, and undefined for all the other elements in X. Therefore, there exist some elements $a \in X$ such that $f(a) \in Y$ (well-defined), and for all other element $b \in X$ we have $f(b)$ is undefined.

Definition 3 [27]: A function f: X → Y is called a *NeutroFunction* if it has elements in X for which the function is well-defined {degree of truth (T)}, elements in X for which the function is indeterminate {degree of indeterminacy (I)}, and elements in X for which the function is outer-defined {degree of falsehood (F)}, where $T, I, F \in [0,1]$, with $(T, I, F) \neq (1,0,0)$ that represents the (Total) Function, and $(T, I, F) \neq (0,0,1)$ that represents the AntiFunction.

Classification of Functions

i) (Classical) Function, which is a function well-defined for all the elements in its domain of definition.

ii) NeutroFunction, which is a function partially well-defined, partially indeterminate, and partially outer-defined on its domain of definition.

iii) AntiFunction, which is a function outer-defined for all the elements in its domain of definition.

Definition 4: A (classical) *Algebraic Structure* (or Algebra) is a nonempty set A endowed with some (totally well-defined) operations (functions) on A, and satisfying some (classical) axioms (totally true) - according to the Universal Algebra.) [26-27-, 28, 29].

Definition 5 ([26, 27]): A (classical) *Partial Algebra* is an algebra defined on a nonempty set PA that is endowed with some partial operations (or partial functions: partially well-defined, and partially undefined). While the axioms (laws) defined on a Partial Algebra are all totally (100%) true.

Definition 6 ([26, 27]): A *NeutroAxiom* (or *Neutrosophic Axiom*) defined on a nonempty set is an axiom that is true for some set of elements {degree of truth (T)}, indeterminate for other set of elements {degree of indeterminacy (I)}, or false for the other set of elements {degree of falsehood (F)}, where $T, I, F \in [0,1]$, with $(T, I, F) \neq (1,0,0)$ that represents the (classical) Axiom, and $(T, I, F) \neq (0,0,1)$ that represents the AntiAxiom.

Classification of Algebras

i) A (classical) *Algebra* is a nonempty set CA that is endowed with total operations (or total functions, i.e. true for all set elements) and (classical) Axioms (also true for all set elements).

ii) A *NeutroAlgebra* (or *NeutroAlgebraic Structure*) is a nonempty set NA that is endowed with: at least one *NeutroOperation* (or *NeutroFunction*), or one *NeutroAxiom* that is referred to the set (partial-, neutro, or total-) operations.

iii) An *AntiAlgebra* (or *AntiAlgebraic Structure*) is a nonempty set AA that is endowed with at least one *AntiOperation* (or *AntiFunction*) or at least one *AntiAxiom*.

Additionally, the PROSPECTOR function is defined in the MYCIN expert system in the following way; it is a mapping from $[-1,1]^2$ into $[-1,1]$ with formula, [10-12]:

$$P(x, y) = \frac{x+y}{1+xy} \qquad (1)$$

This function is a uninorm with neutral element 0, thus it fulfils commutativity, associativity, and monotonicity. $P(-1,1)$ and $P(1,-1)$ are undefined.

3. Results

This section is dedicated to expose the method we will use and the achieved results.
First of all, for convenience we extend $P(x,y)$ to $\bar{P}(x,y)$ such that:
$\bar{P}(x,y) = P(x,y)$ for all $(x,y) \in [-1,1]^2 \setminus \{(-1,1),(1,-1)\}$,
$\bar{P}(-1,1) = \bar{P}(1,-1) = undefined$,
$\bar{P}(undefined, undefined) = undefined$.

$$\bar{P}(undefined, x) = \bar{P}(x, undefined) = \begin{cases} \text{undefined, if } x > 0 \\ x, \text{if } x \le 0 \end{cases}.$$

Definition 7: Let S be a finite set defined as $S = \{(x,y): x, y \in \{k, undefined\},\ k \in \mathbb{Z} \cap [-10,10]\}$.

The operator $\odot$ is defined for every $(x,y) \in S$, such that:

1. If $\bar{P}(x,y)$ is not undefined, then $\odot\ y = \frac{round(\bar{P}(x,y)*10)}{10}$, where *round* is the function that output the interger nearest to the argument.
2. If $\bar{P}(x,y)$ is undefined then $x \odot y = undefined$.

   Then $\odot$ is a finite NeutroAlgebra. This is because $\odot$ is commutative and associative for the subset of elements of S without any undefined component, but it is not associative otherwise.

E.g., if $a = -0.9$, $b = 0.8$, $c = undefined$, then $a \odot (b \odot c) = a$ and $(a \odot b) \odot c = -0.4 \neq a$, therfore associativity is a NeutroAxiom.

We used the function *round* for guarantying $\odot$ is an inner operator.

This research was approved by the SSMSO ethics committee on November 8, 2018, the interviews were conducted during January 2019 after explanation of the research and signing of the informed consent. To evaluate the access barriers, those defined by the Government of Chile (2011) were used, [4], which correspond to four variables.

Variables that have been used in different researches on access to health in Latin America and Chile ([37]) are the following:

1. Geographical and transport access barriers,
2. Cultural access barriers,
3. Financial access barriers,
4. Legal access barrier.

The proposed method is the following:

1. For each previous aspect, the opinion of 28 selected migrants is collected. They are asked to rate each aspect on a scale of 0 to 10 if they have a favorable or neutral opinion about access to public health from the point of view of the access that is measured. On the other hand, they are asked to evaluate on a scale of -10 to -1 if they have an unfavorable opinion.

Let us denote by v_{ij}, $(i = 1,2,\dots,28;\ j = 1,2,3,4)$ the evaluation of the ith migrant on the jth a pect.

2. The value obtained in the evaluation of each aspect for each migrant is rescaled to the interval $[-1,1]$, dividing by 10. That is, $nv_{ij} = \frac{v_{ij}}{10}$ is obtained.
3. It is decided on two different situations:

3.1. If less than 33.333% of the respondents show contradictory results for each fixed j, that is, if there are 4 pairs or less of values $(-1,1)$ or $(1,-1)$, these values are eliminated for aggregating.

3.2. Otherwise the jth aspect is evaluated as "undefined" and it should be reviewed in more detail why there is such a contradiction.

4. When we have the case 3.1. the aggregation of the remaining values is calculated by using $\odot$.

The results obtained from applying this method were as follows:

Table 1 summarizes the assessments provided by the interviewed on the four barriers.

Assessment	Geographical and transport access barriers	Cultural access barriers	Financial access barriers	Legal access barrier
-10	0	0	0	1
-9	0	0	0	2
-8	0	0	0	0
-7	0	0	0	0
-6	0	0	0	4
-5	0	0	3	7
-4	0	0	4	6
-3	0	0	0	0
-2	0	0	0	0
-1	0	0	0	0
0	0	0	0	0
1	0	0	2	3
2	0	0	5	0
3	0	1	1	2
4	2	1	3	3
5	4	2	9	0
6	0	5	0	0
7	1	7	0	0
8	16	8	1	0
9	3	3	0	0
10	2	1	0	0

Table 1: Number of migrants who evaluate the four barriers in the scale -10-10.

Aggregating the data of Table 1 using $\odot$ we have the following results based on Tables 2 and 3 in Appendix A:

1. $\odot_{i=1}^{28} nv_{i1} = 1$, which means there is sufficient evidence that "Geographical and transport access barriers" is good.

2. $\odot_{i=1}^{28} nv_{i2} = 1$, which means there is sufficient evidence that "Cultural access barriers" is good.

3. $\odot_{i=1}^{28} nv_{i3} = -1$, which means there is sufficient evidence that "Financial access barriers" is bad.

4. $\odot_{i=1}^{28} nv_{i4} = -1$, which means there is sufficient evidence that "Legal access barrier" is bad.

For quantitative purpose we calculate the mean of the evaluations for every aspect, which are as follows:

1. $\overline{nv}_{i1} = 7.5$,
2. $\overline{nv}_{i2} = 7.0357$,
3. $\overline{nv}_{i3} = 1.75$,
4. $\overline{nv}_{i4} = -3.4615$.

Conclusion

This paper is dedicated to evaluate the barriers to migrants' access in primary health care in a centre in Chile. We evaluated four types of barriers, which are "Geographical and transport access barriers", "Cultural access barriers", "Financial access barriers", and "Legal access barrier". Twenty-eight migrants of a Family Health Center provided their opinions in a scale from -10 to 10. This is not a statistical study, thus we defined an operator based on the PROSPECTOR function for determining if there is sufficient evidence to evaluate every aspect. We concluded that "Geographical and transport access barriers" and "Cultural access barriers" are good, whereas "Financial access barriers" and "Legal access barrier" are bad. We provided the Cayley table of $\odot$, which is not associative when we included the undefined value and it generates a NeutroAlgebra. We preferred to maintain the undefinition of the PROSPECTOR function because this indicates there is contradiction.

Appendix A

The following tables summarize the Cayley table of the NeutroAlgebra generated by $\odot$.

$x \odot y$	**-1**	**-0.9**	**-0.8**	**-0.7**	**-0.6**	**-0.5**	**-0.4**	**-0.3**	**-0.2**	**-0.1**	**0**
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
-0.9	-1	-1	-1	-1	-1	-1	-1	-0.9	-0.9	-0.9	-0.9
-0.8	-1	-1	-1	-1	-0.9	-0.9	-0.9	-0.9	-0.9	-0.8	-0.8
-0.7	-1	-1	-1	-0.9	-0.9	-0.9	-0.9	-0.8	-0.8	-0.7	-0.7
-0.6	-1	-1	-0.9	-0.9	-0.9	-0.8	-0.8	-0.8	-0.7	-0.7	-0.6
-0.5	-1	-1	-0.9	-0.9	-0.8	-0.8	-0.8	-0.7	-0.6	-0.6	-0.5
-0.4	-1	-1	-0.9	-0.9	-0.8	-0.8	-0.7	-0.6	-0.6	-0.5	-0.4
-0.3	-1	-0.9	-0.9	-0.8	-0.8	-0.7	-0.6	-0.6	-0.5	-0.4	-0.3
-0.2	-1	-0.9	-0.9	-0.8	-0.7	-0.6	-0.6	-0.5	-0.4	-0.3	-0.2
-0.1	-1	-0.9	-0.8	-0.7	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1
un-def.	-1	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0
0	-1	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0
0.1	-1	-0.9	-0.8	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0	0.1
0.2	-1	-0.9	-0.7	-0.6	-0.5	-0.3	-0.2	-0.1	0	0.1	0.2
0.3	-1	-0.8	-0.7	-0.5	-0.4	-0.2	-0.1	0	0.1	0.2	0.3
0.4	-1	-0.8	-0.6	-0.4	-0.3	-0.1	0	0.1	0.2	0.3	0.4
0.5	-1	-0.7	-0.5	-0.3	-0.1	0	0.1	0.2	0.3	0.4	0.5
0.6	-1	-0.7	-0.4	-0.2	0	0.1	0.3	0.4	0.5	0.5	0.6
0.7	-1	-0.5	-0.2	0	0.2	0.3	0.4	0.5	0.6	0.6	0.7
0.8	-1	-0.4	0	0.2	0.4	0.5	0.6	0.7	0.7	0.8	0.8
0.9	-1	0	0.4	0.5	0.7	0.7	0.8	0.8	0.9	0.9	0.9
1	undef.	1	1	1	1	1	1	1	1	1	1

Table 2: Cayley table of $\odot$

$x \odot y$	**un-def.**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1**
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	undef.
-0.9	-0.9	-0.9	-0.9	-0.8	-0.8	-0.7	-0.7	-0.5	-0.4	0	1
-0.8	-0.8	-0.8	-0.7	-0.7	-0.6	-0.5	-0.4	-0.2	0	0.4	1
-0.7	-0.7	-0.6	-0.6	-0.5	-0.4	-0.3	-0.2	0	0.2	0.5	1
-0.6	-0.6	-0.5	-0.5	-0.4	-0.3	-0.1	0	0.2	0.4	0.7	1
-0.5	-0.5	-0.4	-0.3	-0.2	-0.1	0	0.1	0.3	0.5	0.7	1
-0.4	-0.4	-0.3	-0.2	-0.1	0	0.1	0.3	0.4	0.6	0.8	1
-0.3	-0.3	-0.2	-0.1	0	0.1	0.2	0.4	0.5	0.7	0.8	1
-0.2	-0.2	-0.1	0	0.1	0.2	0.3	0.5	0.6	0.7	0.9	1
-0.1	-0.1	0	0.1	0.2	0.3	0.4	0.5	0.6	0.8	0.9	1
un-def.	undef.	undef.	undef.	undef.	undef.	undef.	undef.	undef.	undef.	undef.	undef.
0	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	undef.	0.2	0.3	0.4	0.5	0.6	0.7	0.7	0.8	0.9	1
0.2	undef.	0.3	0.4	0.5	0.6	0.6	0.7	0.8	0.9	0.9	1
0.3	undef.	0.4	0.5	0.6	0.6	0.7	0.8	0.8	0.9	0.9	1
0.4	undef.	0.5	0.6	0.6	0.7	0.8	0.8	0.9	0.9	1	1
0.5	undef.	0.6	0.6	0.7	0.8	0.8	0.8	0.9	0.9	1	1
0.6	undef.	0.7	0.7	0.8	0.8	0.8	0.9	0.9	0.9	1	1
0.7	undef.	0.7	0.8	0.8	0.9	0.9	0.9	0.9	1	1	1
0.8	undef.	0.8	0.9	0.9	0.9	0.9	0.9	1	1	1	1
0.9	undef.	0.9	0.9	0.9	1	1	1	1	1	1	1
1	undef.	1	1	1	1	1	1	1	1	1	1

Table 3: Cayley table of $\odot$ (Continuation).

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Modeling Customer Lifetime Value Under Uncertain Environment.

Eman AboElHamd[1], Mohamed Abdel-Basset[2], Hamed M. Shamma[3], Mohamed Saleh[4], Ihab El-Khodary[5]

[1] Department of Operations Research and Decision Support, Cairo University; e.aboelhamd1@gmail.com
[2] Faculty of Computers and Informatics, Zagazig University; mohamed.abdelbasset@fci.zu.edu.eg
[3] Department of Management, School of Business, The American University in Cairo; shamma@aucegypt.edu
[4] Department of Operations Research and Decision Support, Cairo University; m.saleh@fci-cu.edu.eg
[5] Department of Operations Research and Decision Support, Cairo University; e.elkhodary@fci-cu.edu.eg
* Correspondence: e.aboelhamd1@gmail.com

Abstract: Customer lifetime value (CLV) is an essential measure to determine the level of profitability of a customer to a firm. Customer relationship management treats CLV as the most significant factor for measuring the level of purchases and, consequently, the profitability of a given customer. This motivates researchers to compete in developing models to maximize the value of CLV. Dynamic programming models in general—and the Q-learning model specifically—play a significant role in this area of research as a model-free algorithm. This maximizes the long-term future rewards of a certain agent, given their current state, set of possible actions, and the next state of that agent, assuming the customer represents the agent and CLV is their future reward. However, due to the stochastic nature of this problem, it is inaccurate to obtain a single crisp value for Q. In this paper, fuzzy logic and neutrosophic logic shall be utilized to search for the membership values of Q to capture the stochasticity and uncertainty of the problem. Both fuzzy Q-learning and neutrosophic Q-learning were implemented using two membership functions (i.e., trapezoidal, and triangular) to search for the optimal Q value that maximizes the customer's future rewards. The proposed algorithms were applied to two benchmark datasets: The Knowledge Discovery and Data Mining (KDD) cup 1998 direct mailing campaign dataset and the other from Kaggle, related to direct mailing campaigns. The proposed algorithms proved their effectiveness and superiority when comparing them to each other or the traditional deep Q-learning models.

Keywords: Customer Lifetime Value; Fuzzy Logic; Neutrosophic Logic; Q-Learning; Dynamic Programming; Uncertainty

1. Introduction

Customer lifetime value (CLV) is a crucial concept in customer relationship management (CRM). It is defined as the present value of all future profits that can be obtained from the customers over their lifetime of relationship with a specific firm (as presented in Figure 1). In short, direct marketing is about treating customers differently based on their level of profitability, and CLV is the most reliable indicator in direct marketing for measuring the profitability of customers [4, 20, 27]. CLV depends on many factors including customers' retention rate, acquisition rate, probability of churn, and Recency, Frequency, Monetary (RFM) values [10]. Many researchers competed in developing models that measure CLV [8]. Meanwhile, due to the effectiveness of CLV in determining the level of profitability of the customer, the researchers devoted more interest in the models that maximize the values of CLV, to help the firm in maximizing the long-term profitability of its customers and treat those customers accordingly [28].

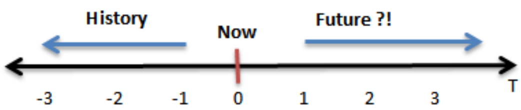

Figure 1. Historical and future periods for CLV

Q-learning model alongside deep learning models proved superiority in this area of research, by helping in maximizing CLV [31, 19]. Meanwhile, those models have a major drawback of overestimating the action values, and hence, recommending unrealistic actions. The rest of this section will be devoted to illustrating the main algorithms of the proposed models. Starting from Q-Learning, passing by Fuzzy logic and Neutrosophic logic; demonstrating the relationship between Fuzzy logic and Neutrosophic logic, and presenting their main ideas and applications. Q-Learning is an off-policy reinforcement learning algorithm that helps in approximating an optimal action to be taken, for the sake of maximizing the long-term reward given the current state. It is considered as an off-policy algorithm, as the Q function learns from actions outside the current policy (i.e., taking random actions) and this is why a policy is not needed. Traditionally, Q-learning was performed by constructing a Q-table that is a matrix of (states and actions) initialized to Zeros as demonstrated in Fig.2.

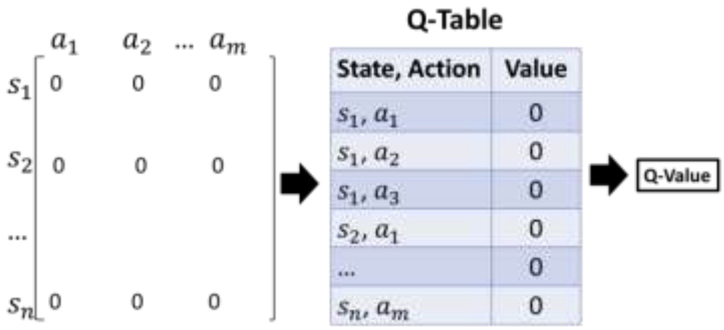

Figure 2. Q-table

In the Q table, states are on rows and actions are on columns and the goal is to select the action that gives the maximum Q value, and consequently the maximum long-term reward. Eq. (1) illustrates the relationship between Q value, current state, current action, next state, and immediate reward mathematically. Where Q(s_t,a_t) is the current Q value, r_t is the reward, Q(s_{t+1},a_t) is the expected reward from the action in the next state, ŋ is a learning rate, finally, γ is the discount factor.

$$Q(s_t, a_t) = Q(s_t, a_t) + ŋ[r_t + \gamma max_a Q(s_{t+1}, a) - Q(s_t, a_t)] \qquad (1)$$

Recently, the researchers integrated Q-learning with deep learning. In this integration, Q values are estimated using deep neural networks [31]. The latter empowers reinforcement learning especially in large and complex problems where finding an optimal solution is impossible, as DQN helps in finding the approximate solution for Q that helps in generalizing the results, as shown in Eq. (2); where $\mathbf{r_t}$ represents the immediate reward and Q(s_t, a_t) is the optimal Q value.

$$Q(s_t, a_t) = r_t + \gamma\, max_a Q^*(s_{t+1}, a) \qquad (2)$$

A fuzzy set is a special kind of sets whose elements have degrees of membership [33]. It differs from the classical set theory, as the latter assumes that the elements of a set have binary degrees of membership, and they either belong to this set or not [35]. This is why it is called the “Crisp” set. Meanwhile, in the Fuzzy set theory, the elements have a real-valued membership function, they belong to this set by a fractional value μ(x), where μ(x)∈[0,1]. Fig.3 demonstrates the difference between the membership function of crisp and fuzzy sets [38]. There are many types of fuzzy membership functions, including Triangular, Trapezoidal (i.e. will be discussed in detail in Section-2.2.1.), Sigmoid, and many others [17]. Fuzzy logic is used in many applications, including decision making, clustering, linguistics, and many more domains where the information is incomplete or imprecise. However, it is rarely used in marketing applications and tested on benchmark datasets alongside Q-learning. Although, it is expected to achieve superior results for generating optimal long term Q values, by relaxing the crisp Q value to different stochastic membership functions (i.e. triangular, trapezoidal, sigmoid, … etc.)

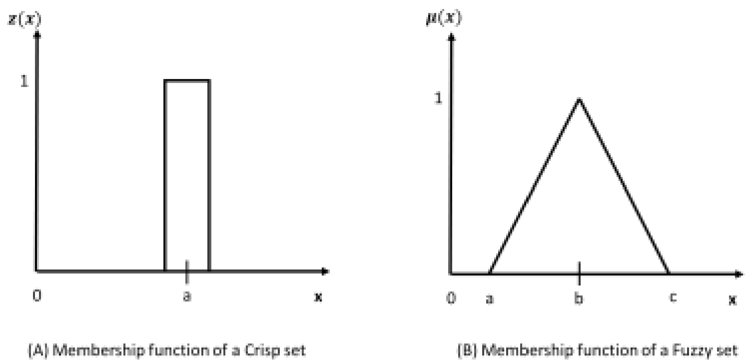

Figure 3. Membership function of crisp and fuzzy sets

Finally, the term Neutrosophic means neutral thought knowledge. It is a combination of two terms (Neuter) and (Sophia), wherein Latin Neuter means “Neutral” and Sophia means “Wisdom”. In general, Neutrosophic set and logic are generalizations of classical fuzzy and intuitionistic fuzzy [40], while neutrosophic Probability and Statistics are generalizations of classical and imprecise probability and statistics [3]. Neutrosophic Logic (NL) is a framework for the unification of many existing logics, such as fuzzy logic, paraconsistent logic, intuitionistic logic, etc. [34, 37]. The main idea of NL is to characterize each logical statement in a 3D-Neutrosophic space, where each dimension of that space represents respectively the truth (T), the indeterminacy (I), and the falsehood (F) of the statement under consideration; where T, I, and F are standard or non-standard real subsets from]0, 1[with not necessarily any connection between them [2]. Many examples can be represented only by neutrosophic logic and neither by fuzzy, nor intuitionistic fuzzy. One of those examples is “Voting” [36]. In general, the neutrosophic set depends on three membership functions (T, I, and F). These functions are independent, and their sum does not add up to 1. Meanwhile, it should add up to 3 [39]. Neutrosophic logic is considered a bigger umbrella of Fuzzy logic. Also, it has many applications however it has not been used so far alongside Q-learning. Although, by combining it with Q-learning, more realistic, and flexible long-term values for Q are expected to be obtained.

Due to the significance of CLV and the effectiveness of Q-learning, fuzzy logic, and neutrosophic logic algorithms; many researchers compete in developing models to utilize these algorithms separately in the marketing context. Meanwhile, each of their implementations has a certain drawback. For instance, neutrosophic logic is not applied yet in a real-life marketing context to maximize CLV [11]. Also, fuzzy logic is not utilized to maximize CLV, but for many other purposes, including clustering the customer base according to their level of profitability to the firm or also measuring it with RFM values instead of CLV [28, 6]. Finally, Q-learning has been combined with different machine learning and deep learning algorithms for that purpose. For instance, some researchers utilized deep learning to predict the optimal value of Q that maximized the long-term profitability of the customers within the firm [31, 19]. Meanwhile, these algorithms overestimated the action values of Q, hence, generated unrealistic actions [14].

Consequently, this paper proposes two models, Fuzzy Q-learning (FQL), and Neutrosophic Q-learning (NQL). The former combines fuzzy logic with Q-learning, to search for an optimal Q value that maximizes long term future rewards. Also, neutrosophic logic is utilized for the same purpose. Both models are implemented using two types of membership functions (triangular, and trapezoidal). Each of them is applied to two benchmark datasets. Also, both of those models are expected to overcome the limitations of the traditional models that overestimate the action values and hence, generate unrealistic actions. The proposed models expected to overcome this by capturing the stochastic nature of the problem through recommending fuzzy membership values for Q instead of crisp ones, in case of fuzzy logic, and replacing these crisp Q values with the neutrosophic three-values membership (T, I, F) in case of neutrosophic logic. The rest of the paper is organized as follows; Section-2 lists the work that is related to this point of research. Section-3 presents the proposed models; while the datasets and the experimental results are presented in Section-4. Section-5 lists the managerial implications of the proposed algorithms, while Section-6 mentions the limitations of this research and future research directions. Finally, Section-7 concludes the proposed work.

2. Background and Related Work

The following two subsections present the related work of utilizing Fuzzy sets, and Neutrosophic sets in the field of machine learning and decision making to empower marketing decisions. The main focus of those sections is the illustration of triangular and trapezoidal membership functions, while the last subsection illustrates the power of Q-learning as a dynamic programming approach in the area of maximizing customer lifetime value.

2.1. Fuzzy Models

Fuzzy sets and Fuzzy logic are attractive area of research for many researchers, to be utilized in the area of CLV. A proposal for customer segmentation using fuzzy c-means clustering and customer ranking using an optimized version of fuzzy AHP has been done [28]. Their proposed model was applied to a large IT company in Iran and proved its effectiveness in grouping the customer base into nine segments. One of their limitations was that they applied their proposed model to only a single industry. Hence, its results were not generalized. Other researchers proposed six fuzzy key performance indicators to measure customer retention and loyalty. They concluded with the effectiveness of these indicators in determining the retention and loyalty of the customers. On top of their limitations was that their study had a limited number of respondents and from a particular management level, for a certain segment from a particular company [32]; while other researchers applied a fuzzy linguistic model that related customer segmentation with campaign activities for more interpretability to the results. Their work was well presented with many implementation details. Meanwhile, it was applied to a single company without generalization, besides the fact that the segmentation based on RFM usually causes a lake of precision [6]. This is a bit different from the work in [5], which added the "Length" dimension to RFM values in their LRFM model, and hence, considered customer loyalty. They calculated the length as (the number of days from the first to the last visit date in a given period). They also performed clustering analysis using LRFM. The main drawback of their work was that most of the illustrative charts are not clear. Others proposed a modeling framework algorithm that estimated the class conditional density functions of Bayesian decision theory for the discrete values, using frequency probability, this stemmed from a set of statistically independent simulations. Meanwhile, for the continuous variables, they assigned a fuzzy logic. Their model outperformed the traditional risk scorecards. Their idea was well presented, however, its applicability was not ensured, as they did not mention any experimental results although it was mentioned that this was a practical approach [24]. Another contribution was an interval type-2 fuzzy model for the quality of web service. Their proposed model showed a greater capability and outperformance over the traditional fuzzy sets in managing the uncertainty of the problem [13]. Their well-structured work would be much more valuable and effective if it was applied to many other industries. Table 1 summarizes the work of applying fuzzy logic in the field of customer lifetime value.

2.2. Neutrosophic Models

In this section, the neutrosophic Q-learning model is presented. Two types of membership functions for the NQL model are illustrated (Trapezoidal and Triangular). The goal is to utilize the neutrosophic model to learn the optimal Q value that maximizes long term rewards. The stochastic nature of the problem is captured by assuming three values for Q (i.e. T, I, and F) instead of a single value, each of which follows the Trapezoidal or Triangular membership function illustrated in the upcoming sub-sections.

2.2.1. Trapezoidal Neutrosophic Q-Learning

In light of neutrosophic logic's definition mentioned in Section-1, which depends upon 3 core values (T, I, and F); this section illustrates how to calculate these values, and how to calculate the model performance measurements [17].

Let H be a universal set, hence, a single-valued neutrosophic set B in H is calculated in Eq. (3)

$$B = \{h, < T_B(h), I_B(h), F_B(h) > | h \in H\}, \tag{3}$$

Where truth membership function (T_B (h)), indeterminacy membership function (I_B (h)), and falsity membership function (F_B (h)) satisfy the following conditions:

$$T_S(z) = \begin{cases} \frac{(z-k)t_S}{(l-k)}, & k \le z < l \\ t_S, & l \le z \le m \\ \frac{(n-z)t_S}{(n-m)}, & m < z \le n \\ 0, & otherwise \end{cases} \tag{4}$$

$$I_S(z)=\begin{cases}\frac{(l-z)+(z-k')i_S}{(l-k')}, & k'\le z<l\\ i_S, & l\le z\le m\\ \frac{z-m+(n'-z)i_S}{(n'-m)}, & m<z\le n'\\ 1, & otherwise\end{cases} \quad (5)$$

$$F_S(z)=\begin{cases}\frac{(l-z)+(z-k'')f_S}{(l-k'')}, & k''\le z<l\\ f_S, & l\le z\le m\\ \frac{z-m+(n\prime-z)f_S}{(n''-m)}, & m<z\le n''\\ 1, & otherwise\end{cases} \quad (6)$$

Where S is a trapezoidal neutrosophic number, $\boldsymbol{k,l,m,n} \in \boldsymbol{R}$. Then $\boldsymbol{S}=([\boldsymbol{k,l,m,n}];\ \boldsymbol{t_s,i_s,f_s})$ is called trapezoidal neutrosophic number (TrNN); and it has one of three possibilities (Positive TrNN, negative TrNN, or normalized TrNN). $\boldsymbol{m}$ is called positive TrNN, if $\boldsymbol{0\le k\le m\le n}$. While, if $\boldsymbol{k\le l\le m\le n\le 0}$, then $\boldsymbol{S}$ is called negative TrNN. If $\boldsymbol{0\le k\le l\le m\le n\le 1}$ and $\boldsymbol{T_s, I_s, F_s} \in [\boldsymbol{0,1}]$, then $\boldsymbol{X}$ is called normalized TrNN. The membership function is demonstrated in Fig.4.

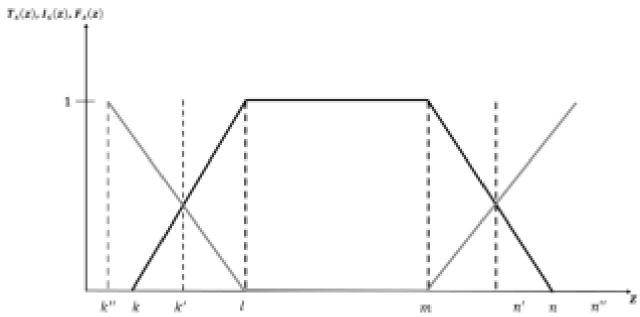

Figure 4. TrNN membership function for truth, indeterminacy, and falsity functions

2.2.1. Triangular Neutrosophic Q-Learning

Assume E is a universe, the triangular neutrosophic number $\bar{\boldsymbol{a}}$ for every z ∈E as ((a, b, c); $\boldsymbol{w_{\bar{a}}}$, $\boldsymbol{u_{\bar{a}}}$, $\boldsymbol{y_{\bar{a}}}$) and the truth, indeterminacy, and falsity membership functions are defined in Eq. (7, 8, and 9) respectively, and demonstrated in Fig.4 . The vector $\bar{\boldsymbol{a}}$ takes one of two forms, if $a \ge 0$ and $a<b<c$, then $\bar{\boldsymbol{a}}$ is called a positive triangular neutrosophic number, while, if $a \le 0$ and $a>b>c$ then $\bar{\boldsymbol{a}}$ is called a negative triangular neutrosophic number [1].

$$T_{\bar{a}}(z)=\begin{cases}\frac{(z-a)w_{\bar{a}}}{(b-a)}, & a\le z<b\\ w_{\bar{a}}, & z=b\\ \frac{(c-z)w_{\bar{a}}}{(c-b)}, & b<z\le c\\ 0, & otherwise\end{cases} \quad (7)$$

$$I_{\bar{a}}(z)=\begin{cases}\frac{(b-z+(z-a)u_{\bar{a}})}{(b-a)}, & a\le z<b\\ u_{\bar{a}}, & z=b\\ \frac{z-b+(c-z)u_{\bar{a}}}{(c-b)}, & b<z\le c\\ 1, & otherwise\end{cases} \quad (8)$$

$$F_{\bar{a}}(z) = \begin{cases} \frac{(b-z+(z-a)y_{\bar{a}})}{(b-a)}, & a \le z < b \\ y_{\bar{a}}, & z = b \\ \frac{z-b+(c-z)y_{\bar{a}}}{(c-b)}, & b < z \le c \\ 1, & otherwise \end{cases} \quad (9)$$

There are three main performance measurements to evaluate the output of the trapezoidal neutrosophic set [9]. These are score function $\boldsymbol{Sc}$, accuracy function Ac, and certainty function E. Assuming a neutrosophic function (g), these measures can be stated as demonstrated in Eqs. (10, 11, and 12).

$$Sc(g) = \frac{2+T_g-I_g-F_g}{3}, \quad (10)$$

$$Ac(g) = (T_g - F_g), \quad (11)$$

$$E(g) = T_g, \quad (12)$$

Based on the score function mentioned in Eq. (10) that was also stated in the work in [22], classified the score of single-valued neutrosophic sets to three major zones (Highly Acceptable Zone, Tolerable Acceptable Zone, and Unacceptable Zone). The three zones and their corresponding intervals are demonstrated in Fig.5. Ranking these scores in descending order helps in selecting the most effective and significant attributes in the decision marking problem at hand. While, the accuracy values range from [-1, 1].

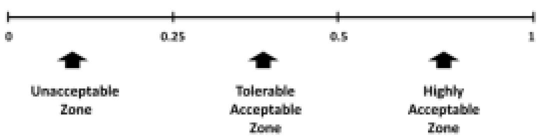

Figure 5. Score zones of single-valued neutrosophic set.

The score and accuracy in Eqs. (10, and 11) are very essential in determining the ranking of the set of alternatives at hand. The score and accuracy of each neutrosophic numbers (x, y) are compared, as mentioned in Eqs. (13, 14, and 15)

$$If\ \mathrm{Sc(x)} > \mathrm{Sc(y)}\ Then\ x > y \quad (13)$$
$$If\ \mathrm{Sc(x)} = \mathrm{Sc(y)}\ \&\ Ac(x) > Ac(y)\ Then\ x > y \quad (14)$$
$$If\ \mathrm{Sc(x)} = \mathrm{Sc(y)}\ \&\ Ac(x) < Ac(y)\ Then\ x < y \quad (15)$$

One of the most significant steps in neutrosophic logic is the concept of de-neutrosophication [9]. In this step, the three neutrosophic values (T, I, F) are converted to a single crisp value using Eq. (16)

$$\Psi = 1 - \sqrt{\frac{(1-T_x)^2 + {I_x}^2 + {F_x}^2}{3}} \quad (16)$$

Researchers utilized neutrosophic logic in many domains including Physics [29], speech recognition [26], supply chain [12], or in decision-making process that is much more relevant to the work in [25]. Other researchers proposed an approach to the binary classification problem using ensemble neural networks based on interval neutrosophic set and bagging technique. They built two neural networks to predict the degree of truth and falsity membership values and estimated the degree of indeterminacy. Their proposed algorithm was tested on three benchmark datasets on UCI and proved its superiority over the single pair of neural networks. Their work was well presented, meanwhile, only applied to a medical application, and it is recommended to be applied to other areas to test its robustness [17]. While others utilized neural networks to a bit similar mechanism but for a multi-class classification task. They could provide an assessment for the uncertain predicted values by utilizing two neural networks also to predict the true and false membership values. The indeterminacy value was estimated as well. Their proposed algorithm was tested on different benchmark datasets from UCI and expected to be applied on a real-life "oil and gas" dataset. Meanwhile, their work was not compared to other work to prove its superiority [18]. On the other hand, there was another contribution of single-valued neutrosophic set logic in data mining tasks including neutrosophy decision trees, neutrosophy prototypes, and neutrosophy clustering. They proposed a novel way to calculate the score of the alternatives at the multi-criteria decision-making problem. Meanwhile, their proposed model was not applied to a real-life dataset to test its robustness and effectiveness [22].

Finally, one of the well-structured and well-organized literature survey papers in neutrosophic was written by other researchers who related machine learning tasks to neutrosophic logic and mentioned the contribution of the research in each research direction. For instance, how neutrosophic set dramatically enhanced the traditional clustering techniques and prediction models. They concluded their paper with the fact that relating neutrosophic to Q-learning and deep learning is an untouched research direction, and this is one of the motivations of this paper [11]. The major contribution of neutrosophic set logic in machine learning is listed in Table 1. Meanwhile, none of the work in the literature tackled the problem of maximizing CLV using neutrosophic logic or neutrosophic Q-learning, and this is one of the main contributions of this research.

2.2. Q-Learning Models

Q-learning is a very reliable and robust model-free algorithm, that has been applied in many research areas; either standalone or alongside other optimization algorithms (i.e. Artificial Neural Network (ANN) or Deep Learning) to enhance its performance and experimental results, through generalizing its results. Other researchers tried to utilize Q-learning to solve the model-free optimal tracking control problem. They approximated the Q function using ANN. Their method showed superiority over traditional exploration methods. Although they tested the effectiveness of their proposed algorithm on a set of simulation studies, it would have been much more informative, if they tested it on real-life datasets [21]. Others combined policy gradient with an off-policy Q-learning. Hence, they could estimate the Q values from the action preferences of the policy. Their model showed outperforming results when it was tested on a set of numerical examples and Atari Games. However, it was not tested in real-life industrial applications [23].

Other researchers explored how DQN could be used to predict CLV in video games. To test their model, they compared the performance of DQN to parametric models (i.e. Pareto/NBD) and it outperformed it [7]. Similar to the research of this manuscript are two publications [31, 19]. The former proposed a framework that utilized deep Q networks to accomplish two major contributions [31]. First, introducing a modified version of RFM value that can be used to define the state space of the donors; meanwhile, FRM values are ambiguous, and using a deterministic nature problem setup is inappropriate. Second, they tried to determine the optimal marketing action in both discrete and continuous action spaces. They applied their proposed algorithm to the KDD cup 1998 mailing dataset. The researchers in [19] built on their work. They worked on the same dataset and with the same algorithm but had a set of differences. The latter utilized deep learning mainly to learn the representation of the states in a partially observable environment. Furthermore, they proposed a hybrid approach that combined supervised learning to learn the hidden states and reinforcement learning to select the optimal action [19]. Yet, their proposed algorithms had the main limitation of overestimating the action values and consequently, resulted in unrealistic actions; and this is the major drawback of combining deep learning with reinforcement learning [14]; and the main motivation of the work of this paper. The major contributions of applying reinforcement learning in CLV are summarized in Table 1.

Table 1 Major contribution of reinforcement learning in CLV

Publication's Title	Proposed Algorithm	Reference
Model-free optimal tracking control via critic-only Q-learning	Reinforcement Learning	[21]
Combining policy gradient and Q-learning	Reinforcement Learning	[23]
Autonomous CRM control via CLV approximation with deep reinforcement learning in discrete and continuous action space	Reinforcement Learning	[31]
Recurrent reinforcement learning: a hybrid approach	Reinforcement Learning	[19]
Customer lifetime value in video games using deep learning and parametric models	Reinforcement Learning	[7]
Machine learning in Neutrosophic Environment: A Survey	Neutrosophic Logic	[11]
Role of neutrosophic logic in data mining	Neutrosophic Logic	[22]
Ensemble neural networks using interval neutrosophic sets and bagging	Neutrosophic Logic	[17]
Multiclass classification using neural networks and interval neutrosophic sets	Fuzzy Logic	[18]
Customer lifetime value determination based on RFM model	Fuzzy Logic	[28]
Fuzzy indicators for customer retention	Fuzzy Logic	[32]
A Fuzzy Linguistic RFM Model Applied to Campaign Management	Fuzzy Logic	[6]
New Approach for Customer Clustering by Integrating the LRFM Model and Fuzzy Inference System	Fuzzy Logic	[5]
Consumer credit limit assignment using Bayesian decision theory and Fuzzy Logic–a practical approach	Fuzzy Logic	[24]
An interval type-2 fuzzy model of compliance monitoring for quality of web service	Fuzzy Logic	[13]

Each of the algorithms applied by other researchers has a set of advantages and disadvantages. Table 2 lists some of them. Combining those algorithms in the proposed algorithm avoids their disadvantages and tries to make the best out of their advantages.

Table 2: Advantages and Limitations of the traditional techniques

Algorithm	Advantages	Limitations
Reinforcement Learning	1. Able to solve very complex problems. 2. Can correct the errors that occurred during the training process. 3. In the absence of a training dataset, it can learn from its experience. 4. In outperforms humans in many tasks [30] 5. Achieves the ideal behavior of the model while mainlining the balance between exploration and exploitation	1. It needs a lot of data. 2. Needs a lot of computations. 3. It assumes the world is Markovian, which is not always the case. 4. To obtain the best of it, one can combine it with other algorithms (i.e., Deep learning)
Fuzzy Logic	1. Can be used to solve complex problems 2. The structure of it is easy and understandable 3. It can offer accurate and acceptable reasoning 4. Deal with uncertainty	1. Setting "exact" fuzzy rules and membership functions are difficult tasks 2. Its results are not always accurate 3. Expensive validation and verification 4. Doesn't support real-time response
Neutrosophic Logic	1. An effective way to handle antinomies or uncertainties 2. Indeterminacy plays an essential role in NL, while it's ignored in other methods 3. Based on the above two advantages, NL has more ability to assess cause-effect relationships 4. Perfectly handle the situations that contain incomplete information	1. Although NL proved its effectiveness in many cases, it might have some limitations in its applicability in a few real- life case studies

2.3. Customer Lifetime Value

Data mining played a significant role in measuring CLV. Meanwhile, traditional data mining techniques mainly tried to segment the customers according to their CLV [16], classified them accordingly, or even identified the potential of risky customers [40]. Yet, those contributions indirectly supported the business decision. Thus, this research aims to close the decision-making process loop, by utilizing reinforcement learning techniques (i.e. Q-learning) alongside stochastic programming methods (Fuzzy logic and Neutrosophic logic) to provide actions that directly contribute to maximizing the CLV of the customers. The closest contributions to the work of this paper are [31, 19]. Each of them utilized Q-learning for the same purpose. They trained machine learning [31] or deep learning [19] algorithms to learn the Q value. The proposed models of this paper integrate Q-learning with either fuzzy logic or neutrosophic logic instead of deep learning models. This is expected to generalize the Q values, generate realistic actions, and overcome the overestimation issue caused by learning the Q values using deep learning algorithms. The proposed models are illustrated in more detail in Section-3.

3. Proposed Models

This section presents two proposed models; one of them is a novel one, that combines neutrosophic logic with Q-learning. The other model combines fuzzy logic with Q-learning. The latter is not considered as a novel model, yet its implementation in a marketing context on two benchmark datasets including the Paralyzed Veterans of America (PVA) dataset, and Kaggle direct marketing dataset^1 is its source of novelty. Each of these two models is applied using two membership functions (i.e. Triangular and Trapezoidal). Both of the proposed models are applied to two datasets as will be illustrated in the following subsections.

3.1. Fuzzy Q-Learning

In Fuzzy Q-Learning (FQL) the goal and/or the constraints are fuzzy, however, the system under control is not necessarily being fuzzy [15]. FQL(s_t, a_t) estimates the value of taking action a at state s at a certain time t. The value of the state s is defined as the optimal state-action pair, as demonstrated in Eq. (17). Hence, FQL is a combination of the immediate rewards plus the discounted value of the next state $s_{t+1}$and the constraints on selecting the action $a_t$in state s_t, as illustrated in Eq. (18), while Eq. (19) demonstrates the update rule of FQL. Algorithm-1 lists the main steps of the FQL algorithm, assuming γ is the discount factor, and ŋ is the learning rate.

$$V(s) = Max_a FQL(s_t, a_t) \quad (17)$$

$$FQL(s_t, a_t) = E[(r_t + \gamma V(s_{t+1})) \wedge \mu_c(s_t, a)] \quad (18)$$

$$\Delta FQL(s_t, a_t) \leftarrow ŋ[(r + \gamma V(s_{t+1})) \wedge \mu_c(s_t, a_t) - FQL(s_t, a_t)] \quad (19)$$

Algorithm 1: FQL

Step-1: Input γ and ŋ where $\gamma \in [0, 1]$ and ŋ $\in [0, 1]$

Step-2: Initialize FQL values

$$FQL(s_t, a_t) \leftarrow 0$$

Step-3: Until FQL values converge do

{

3.1. $s_t \leftarrow$ current state

3.2. Select action (a) with the highest FQL (*if multiple exist, select one of them randomly*)

3.3. Apply action (a) and observe the new state (s_{t+1}) and a reward (r_t)

3.4. Update Eqs. (21, 22)

$$FQLnew = ŋ[(r_t + \gamma V(s_{t+1})) \wedge \mu_c(s_t, a_t) - FQL(s_t, a_t)] \quad (20)$$

$$FQL(s_t, a_t) \leftarrow FQL(s_t, a_t) + FQLnew \quad (21)$$

}

3.2. Neutrosophic Q-Learning

The idea of the proposed neutrosophic Q-learning (NQL) algorithm is to utilize the three values (T, I, F) of neutrosophic to learn the optimal long term reward of Q. Hence, in short, the process of the proposed algorithm starts with replacing the single value of Q with the three neutrosophic components (T, I, F), calculating the score of each value, then applying de-neutrosophication to convert the results back to a single value to be able to inject it in Eq. (2).

The de-neutrosophication is applied using many techniques, the most popular is either to rank the alternatives based on their score ranges mentioned in Fig.5 or apply Eq. (17). The main goal of utilizing NQL is to capture the stochasticity of the problem in the neutrosophic three values; such that at each state Q is represented by three independent values (true, indeterminate, and false) instead of a single crisp value. This is expected to learn the optimal value of Q without overestimating its action values and consequently, generate reliable proposed actions. The main steps of NQL are listed in Algorithm 2.

Algorithm 2: NQL

Step-1: Input γ and ŋ where $\gamma \in [0,1]$ and ŋ $\in [0,1]$

Step-2: Initialize NQL values

$$Q \leftarrow 0$$

$$NQL(s_t, a_t) \leftarrow 0$$

Step-3: Until NQL values converge do

{

3.1. $s_t \leftarrow current\ state$

3.2. $Calculate\ three\ values\ (T, I, F)$

$If\ Q\ follows\ the\ Trapezoidal\ membership\ function$

$apply\ Eq.(4,5,6)$

$If\ Q\ follows\ the\ Traigular\ membership\ function$

$apply\ Eq.(7,8,9)$

3.3. Calculate the score function using Eq. (11)

Determine the three zones (Highly acceptable, Tolerance acceptable, or Unacceptable), based on the score range of values (mentioned in Fig.5)

Apply de-neurosophication (whether by ranking the attributes or applying Eq. (17))

3.6. $Update\ Q\ value$

$$QCurrState = r_t + \gamma max_a Q(s_{t+1}, a) - NQL(s_t, a_t) \quad (22)$$

$$NQL(s_t, a_t) = NQL(s_t, a_t) + ŋQ\_nextState \quad (23)$$

3.7. Select action (a) with the highest NQL (if multiple exists, select one of them randomly)

3.8. Calculate Q value

$$NQL(s_t, a_t) = r_t + \gamma\ max_a NQL^*(s_{t+1}, a) \quad (24)$$

}

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4. Experimental Results

This section presents the results of the experiments that have been done. Two proposed algorithms (FQL and NQL) are applied to two benchmark datasets. First, KDD cup 1998 direct mailing campaign dataset [19], and the second one is a direct marketing dataset from Kaggle^1. Each algorithm is applied using two different membership functions (i.e. Trapezoidal and Triangular) on different train-test data split types, to test the effectiveness of each algorithm on different data sizes.

4.1. KDD Dataset

The proposed models are applied to the KDD cup 1998 direct mailing campaign dataset [19]. It has been collected by Paralyzed Veterans of America or PVA for short. It is a non-profit organization that has programs and services for United States veterans with spinal cord injuries or diseases. Hence, the training data of this dataset contains a record for every donor who received a PVA donation mailing campaign and didn't make a donation in the last 12 months. It has been collected for 23 distinct periods for a total number of donors of 95,412. It describes whether and how each of them donated as well as their donation amount. It consists of 477 independent variables and two types of dependent variables represent the donation flag and amount. The proposed model is applied to the only subset of these variables to construct the Q-learning tuple (current state, action, next state, and reward). The current state of each donor is assumed to be a five-dimensional vector describes (how recently the donor donated last (r_0), how frequently he donates (f_0), their average donation amount (m_0), how many times PVA sends him an email in the last six months (ir_0), and how many times PVA has sent her emails (if_0)). The next state is also assumed to be a 5-dimensional tuple as $(r_1, f_1, m_1, ir_1, if_1)$, the transaction from a current state to the next state was through taking an action (a). A direct mailing campaign is a well-known task in CRM, where the goal is to decide which mailing type to send to the customer to maximize their long-term profitability (i.e. donation amount). Consequently, the KDD dataset consists of 12 mailing types to choose from (i.e. sending a thank you mail, blank cards, Christmas cards with labels, etc.). The rewards represent the donation amount of each donor, these are range from ($0 to $1000) in the training data. The proposed models of FQL and NQL are implemented using the Python programming language. Table 5 and Table 6 summarize the values of average rewards ($) of the proposed models using different membership functions, different split types of train-test data, and different action selection policies; where the real policy represents the action selection policy stated in the dataset while, uniformly random policy is a selection of the actions based on uniform distribution. This is what was exactly done by other researchers to be able to compare the results of FQL and NQL with the results of their proposed algorithm [19]. Finally, the average reward reported in Table 3 and Table 4 is the average of all Q values at every iteration out of 10 iterations.

Table 3. Avg. Reward ($) of FQL Using Different Train-Test Data Split Type and Different Membership Functions

	Real Policy (R)		Uniformly Random Policy (U)	
Train-Test Data Split	Trapezoidal Function	Triangular Function	Trapezoidal Function	Triangular Function
10 Fold CV	9.98	9.19	9.42	9.48
50-50	9.56	9.43	9.36	9.26
70-30	9.45	8.90	9.55	9.48
80-20	9.54	8.90	9.54	9.49

The results of Table 3 and Table 4 are compared to the results of the researchers in [19] as they utilized DQN on the same dataset and for the same purpose of this study, that is searching for the Q value that maximizes the long-term reward of every donor. During their study, they performed 3 action selection criteria that is whether being uniformly random policy (U), a probability matching policy (M), or a Real Policy (R). Only the results of the former and latter criteria are reported in Table 5, as it matches the same action selection criterion of the proposed models.

Table 4. Avg. Reward ($) of NQL Using Different Train-Test Data Split Type and Different Membership Functions

	Real Policy (R)		Uniformly Random Policy (U)	
Train-Test Data Split	Trapezoidal Function	Triangular Function	Trapezoidal Function	Triangular Function
10 Fold CV	9.45	9.80	9.73	9.27
50-50	9.81	9.25	9.67	9.61
70-30	9.40	9.47	9.52	9.29
80-20	9.48	9.45	9.54	9.25

Table 5. Average rewards of Deep Reinforcement Learning models

Reinforcement Learning Models	Avg. Rewards ($) – "Uniform Policy"	Avg. Rewards ($) – "Real Policy"
DQN	9.44	7.03
RL_RNN	9.65	7.62
RL_LSTM	9.60	7.27
SL-RNN + RL_DQN	9.86	7.80
SL_LSTM + RL_DQN	9.81	7.91

The experimental results reported in Table 3, Table 4, and Table 5 proves the superiority of both Fuzzy Q-Learning and Neutrosophic Q-learning in generating higher average reward values than in the deep reinforcement learning, in the case of real policy action selection and mainly using trapezoidal membership function. Meanwhile, this is not usually the case, in the case of the uniform policy. Table 5 lists the average reward values of deep reinforcement learning under different dataset sizes. While, Table 6, demonstrates the results of FQL under different dataset sizes and different action selection policies vs maximum avg. rewards generated from a deep reinforcement learning method. Comparing the results of Table 3 to the results of Table 4, it is obvious that FQL using 10-fold CV type, generated a slightly higher average reward value than SL-RNN + RL_DQN on a dataset of size 500K; however, this is not the case if Table 4 results are compared with Table 6, and the same finding can be by comparing deep reinforcement learning results in Table 6 to NQL results in Table 7.

Table 6. Average rewards of FQL vs Deep Reinforcement Learning models under different data sizes

	Fuzzy Q-Learning		**Deep Reinforcement Learning Algorithms**
Data Size	Trapezoidal Function	Triangular Function	
50K	9.22	9.38	9.74
100K	9.28	9.14	9.69
200K	9.26	9.34	9.78

Table 7. Average rewards of NQL vs Deep Reinforcement Learning models under different data sizes

	Neutrosophic Q-Learning		**Deep Reinforcement Learning Algorithms**
Data Size	Trapezoidal Function	Triangular Function	
50K	9.50	9.18	9.74
100K	9.26	9.52	9.69
200K	9.63	9.33	9.78

4.2. Kaggle Dataset

This is one of Kaggle's direct mailing campaign datasets. It includes data from one of the direct marketers, who sells his products only via a direct email. The marketer sends catalogs with product characteristics to customers who then order directly from the catalogs. He has developed customer records to learn what makes some customers spend more than others. This dataset includes data for 1000 customers each of which has a set of variables (represent their state in the developed Q learning model) including their age, gender, whether he owns a home or not, their marital status, their location, salary, number of children he has, history of their previous purchases, number of catalogs sent to him, and their purchased amounts ($). The purchasing decision of each customer takes them to the next state that is also represented by these components, while the rewards are the monetary value of the purchases. Both FQL and NQL have been applied to the dataset to optimizing the Q value using either fuzzy logic or neutrosophic logic with their different membership functions (i.e. triangular, and trapezoidal). Fig.6 demonstrates the average rewards of the FQL algorithm generated by each of its membership functions in different cross-validation types. It is obvious that none of the membership functions strictly dominates the other, meanwhile, we can trust the trapezoidal membership function as it generates a higher average reward in (70-30 and 80-20) train-test split types where more training data is provided.

On another hand, NQL is used for the same purpose of optimizing Q value using the same membership functions of triangular, and trapezoidal. Meanwhile, the outperformance of trapezoidal is obvious in NQL and all train-test data split types, as demonstrated in Fig.7.

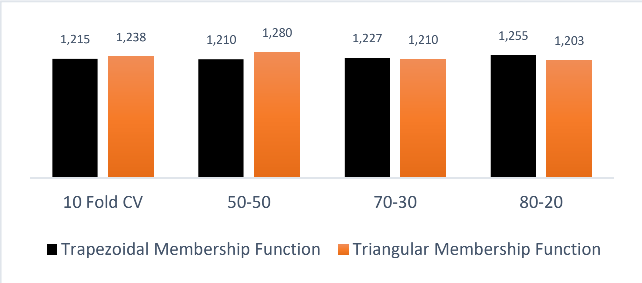

Figure 6 Avg. reward ($) for FQL

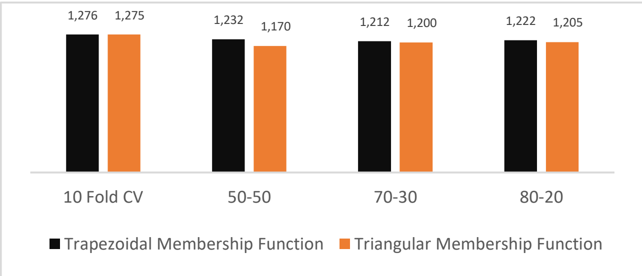

Figure 7 Avg. reward ($) for NQL

5. Managerial Implications

This section presents the managerial implications of the proposed models and how each of them can help in the decision-making process. The proposed models have a set of advantages that boost their flexibility and applicability; including the fact that both depend only on a few parameters, and match the stochastic nature that exists in most real-life situations. Finally, the ease of the implantation of both models, and the possibility of their generalization, promote their applicability in many business situations. Consequently, the proposed models are expected to be of interest to both managers and researchers. The former can apply them on real-life datasets to maximize CLV. While, researchers might apply the proposed models on other datasets to test their robustness, modify them to fill any observed gap or limitation.

6. Limitations and Future Research

This research can be an atom for many future research directions. Meanwhile, it has a set of limitations, including being applied to benchmark datasets, not on real datasets. Moreover, it was not applied to many benchmark datasets to test its robustness and reliability, meanwhile, this is because most of the contributions of the literature review have been done on either hypothetical data or rarely on benchmark datasets, or even built only on a theoretical model. Hence, it was difficult to find many datasets to test the proposed models. Also, it utilized the basic versions of fuzzy logic, neutrosophic logic, and Q-learning models, without contributing to them. Meanwhile, for future research, an advanced version of each algorithm might be applied, for instance, a weighted version of neutrosophic numbers might be utilized [9]. Furthermore, the parameters of Fuzzy and Neutrosophic logic can be optimized using one of the optimization algorithms (i.e. artificial neural network). Another direction is to combine deep reinforcement learning with neutrosophic Q-learning, to avoid the main drawback of overestimating the action values generated from one of the most popular deep reinforcement learning algorithms (i.e. deep Q-learning algorithm). The last but not least research direction is to apply the proposed models on other datasets or applications to test their robustness and reliability.

7. Conclusion

Customer lifetime value (CLV) plays a significant role in determining the value of a customer's profitability within a firm. This motivated the researchers to compete in developing models that maximize CLV. A bunch of those researchers utilized the Q-Learning model for this purpose. They combined Q-learning with deep learning to be able to select the action that would maximize the long term profitability of the customers. In this paper, two models were proposed (Fuzzy Q-Learning and Neutrosophic Q-Learning). The former combined Fuzzy logic with Q-learning while the latter combined Neutrosophic logic with Q-learning. Both models were utilized to select a stochastic value for Q that would maximize the long-term reward, instead of having a single crisp value that may overestimate the action values and make them unrealistic. Two membership values were utilized in each model (i.e. Trapezoidal and Triangular). The proposed models were applied to two different datasets. KDD cup 1998 direct mailing campaign dataset was the first one. While Kaggle direct marketing campaign dataset was the second. The proposed models were applied to both using different data split types and were compared to deep reinforcement learning models in the case of the KDD dataset. The proposed algorithms showed superiority, whether under different action selection criteria or different dataset sizes. The results of FQL and NQL were compared to each other in the case of the Kaggle dataset as it was not utilized in any of the previous research. Trapezoidal membership function generated higher average reward values in most of the cross-validation types, especially, in the case of NQL.

Conflict of Interest/ Competing interests
The authors declare that there is no conflict of interest in the research.

Availability of data and material
This manuscript depends on two benchmark datasets; both are available online and already cited within the paper.

Code availability
Code is available upon request.

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This article does not contain any studies with human participants or animals performed by any of the authors.

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Triangular Neutrosophic Topology

Ali Hamza [1]*, Sara Farooq [1], and Muhammad Rafaqat[2]

[1] Department of Mathematics and Statistics, The University of Lahore, Pakistan; alifm2909@gmail.com
[1] Department of Mathematics and Statistics, The University of Lahore, Pakistan; sarafarooq447@gmail.com
[2] Department of Mathematics and Statistics, The University of Lahore, Pakistan; muhammad.rafaqat@math.uol.edu.pk
Correspondence: alifm2909@gmail.com

Abstract: In this current article, the primary focus is to develop the concept of neutrosophic topology as triangular neutrosophic topology. We extend the neutrosophic set operations, we newly introduce the defuzzification, examples, and basics to clear the concept and new possibilities. Finally, we also investigate some properties such as neutrosophic exterior, neutrosophic subspace, neutrosophic boundary, and neutrosophic closure.

Keywords: Neutrosophic set, Triangular neutrosophic topology, MCDM, neutrosophic interior, neutrosophic exterior, neutrosophic subspace, and neutrosophic boundary.

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1. Introduction

Mathematicians, researchers, and analysts from all over the world work hard to develop new strategies to overcome decision-making problems in a different scenario. It widely uses in medicine, MCDM, engineering, and other math fields. The most challenging issue is to deal with critical situations such as theoretical stuff. To handle vagueness and uncertain situations in both practical and theoretical problems, researchers introduce theories like fuzzy, and neutrosophic sets. The neutrosophic set [1] is based on Truth, indeterminacy, and falsity, it is more effective than crisp and fuzzy.

The idea of triangular neutrosophic numbers was the new development in the line of neutrosophic. Smarandache [2,3] introduce the concept of neutrosophic sets. Un these concepts we have a degree of membership of truth, degree of indeterminacy, and degree of falsity. Many researchers [4-7] do their best and introduce new possibilities such as topology on neutrosophic sets. Other efforts of researchers [8-12] are also remarkable.

We put forward the concept of neutrosophic topology and design new operations and possibilities. MCDM is a wide and developed field and other researchers [13-32] put this concept forward. Triangular neutrosophic numbers plays important role in multi-criteria decision-making, and now it enhances the new evolutions in topology. We also introduce a decent way of defuzzification to make it useable in topological spaces. We also discuss neutrosophic interior, neutrosophic exterior, and other important things in this article.

The idea of neutrosophic is not limited and researchers in recent times proposed, Triangular neutrosophic numbers, Trapezoidal neutrosophic numbers, Pentagonal neutrosophic numbers, Hexagonal neutrosophic numbers, Heptagonal neutrosophic numbers. Moreover, the Octagonal neutrosophic numbers, Nonagonal neutrosophic numbers were presented and publisher by us [33-34]. By using neutrosophic techniques, researchers overcome real life problems [35-39].

2 Preliminaries

The related definitions are given below.

Definition 2.1: We define neutrosophic set $\dot{A}$ over universe of discourse as:

$$\dot{A} = \{\langle \dot{x}, \mu_A(\dot{x}), \sigma_A(\dot{x}), \Upsilon_A(x)\rangle : \dot{x} \in \dot{X}\}$$

Where $\mu_A, \sigma_A, \Upsilon_A : X \to]\dot{0}^-, 1^+[$ and $0^- \leq \mu_A(\dot{x}) + \sigma_A(\dot{x}) + \Upsilon_A(x) \leq 3^+$, the value of neutrosophic sets takes from standard and non-standard of $]0^-, 1^+[$. If we consider the value from real life problem, it will become hard to use value of neutrosophic sets takes from standard and non-standard of $]0^-, 1^+[$. For our convenient we take the value from the subset of [0,1]. $\dot{N}(\dot{X})$.

Definition 2.2: Apply formula $\frac{\acute{a}+\acute{b}+\acute{c}}{3}$ and here, $\dot{A}, \dot{B} \in \dot{N}(\dot{X})$. So,

- (Inclusive) If $\mu_{\dot{A}}(\dot{x}) \leq \mu_{\dot{B}}(\dot{x})$, $\sigma_A(\dot{x}) \geq \sigma_B(\dot{x})$, $\Upsilon_{\dot{A}}(x) \geq \Upsilon_{\dot{B}}(x)$ for every $\dot{x} \in \dot{X}$, then neutrosophic subset can be define as $\dot{A} \sqsubseteq \dot{B}$ where, $\dot{A}$ neutrosophic subset of $\dot{B}$ and then neutrosophic subset can be define as $\dot{B} \sqsubseteq \dot{A}$ where, $\dot{B}$ neutrosophic subset of $\dot{A}$.
- (Equality) if $\dot{A} \sqsubseteq \dot{B}$ and $\dot{B} \sqsubseteq \dot{A}$ then $\dot{A} = \dot{B}$.
- (Intersection) The intersection in neutrosophic sense can be defined as: $\dot{A} \sqcap \dot{B}$ and defined:

  $$\dot{A} \sqcap \dot{B} = \{\dot{x}, \mu_{\dot{A}}(\dot{x}) \wedge \mu_{\dot{B}}(\dot{x}), \sigma_A(\dot{x}) \vee \sigma_B(\dot{x}), \Upsilon_{\dot{A}}(x) \vee \Upsilon_{\dot{B}}(x) : \dot{x} \in \dot{X}\}$$
- (Union) The union in neutrosophic sense can be define as

  $$\dot{A} \sqcup \dot{B} = \{\dot{x}, \mu_{\dot{A}}(\dot{x}) \vee \mu_{\dot{B}}(\dot{x}), \sigma_A(\dot{x}) \wedge \sigma_B(\dot{x}), \Upsilon_{\dot{A}}(x) \wedge \Upsilon_{\dot{B}}(x) : \dot{x} \in \dot{X}\}$$
- (Compliment) The compliment $\dot{A}^c$ in neutrosophic sense can be defined as:

  $$A^{\dot{c}} = \{\langle \dot{x}, \Upsilon_{\dot{A}}(x), 1 - \sigma_A(\dot{x}), \mu_{\dot{A}}(\dot{x})\rangle : \dot{x} \in \dot{X}\}$$
- (Universal set) It can be defined as: $\mu_{\dot{A}}(\dot{x}) = 1, \sigma_A(\dot{x}) = 0, \Upsilon_{\dot{A}}(x) = 0$ for all $\dot{x} \in \dot{X}$.
- (Empty set) It can be defined as: $\mu_{\dot{A}}(\dot{x}) = 0, \sigma_A(\dot{x}) = 1, \Upsilon_{\dot{A}}(x) = 1$ for all $\dot{x} \in \dot{X}$. We can denote it as $\emptyset$.

Example 1: A triangular neutrosophic problem is given below:

$$\dot{A} = \{\langle \dot{x}, (0.2,0.4,0.6)(0.3,0.5,0.8)(0.2,0.8,0.8)\rangle, \langle y, (0.4,0.7,0.2)(0.6,0.7,0.2)(0.4,0.5,0.6)\rangle\}$$

$$\dot{B} = \{\langle \dot{x}, (0.8,02,0.3)(0.4,0.6,0.1)(0.7,0.3,0.5)\rangle, \langle \dot{y}, (0.1,0.2,0.4)(0.4,0.2,0.7)(0.1,0.2,0.4)\rangle\}$$

$$\dot{C} = \{\langle \dot{x}, (0.9,0.3,0.1)(0.2,0.6,0.3)(0.7,0.5,0.6)\rangle, \langle \dot{y}, (0.1,0.9,0.5)(0.2,0.4,0.6)(0.5,0.4,0.9)\rangle\}$$

Apply formula $\frac{\acute{a}+\acute{b}+\acute{c}}{3}$,

Let $\dot{X} = \{\dot{x}, \dot{y}\}$ and $\dot{A}, \dot{B}, C^{\cdot} \in \dot{N}(\dot{x})$ then:

$$\dot{A} = \{\langle \dot{x}, 0.4,0.5,0.6\rangle, \langle \dot{y}, 0.4,0.3,0.5\rangle\}$$

$$\dot{B} = \{\langle \dot{x}, 0.4,0.3,0.6\rangle, \langle \dot{y}, 0.2,0.4,0.3\rangle\}$$

$$\dot{C} = \{\langle \dot{x}, 0.4,0.3,0.6\rangle, \langle \dot{y}, 0.5,0.4,0.6\rangle\}$$

We have $A^{\cdot} \sqsubseteq \dot{B}$.

Neutrosophic union of $\dot{B}$ *and* $\dot{C}$ as:

$$\dot{B} \sqcup \dot{C} = \{\langle \dot{x}, (0.4 \vee 0.4), (0.3 \wedge 0.3), (0.6 \wedge 0.6)\rangle, \\ \langle y, (0.\dot{2} \vee 0.5), (0.4 \wedge 0.4), (0.3 \wedge 0.6)\rangle\}$$

$$\dot{B} \sqcup \dot{C} = \{\langle \dot{x}, 0.4,0.3,0.6\rangle, \langle \dot{y}, 0.5,0.4,0.3\rangle\}$$

The intersection in neutrosophic sense of $\dot{A}$ and $\dot{C}$

$$\dot{A} \sqcap \dot{C} = \{\langle \dot{x}, (0.4 \wedge 0.4), (0.5 \vee 0.3), (0.6 \vee 0.6)\rangle, \\ \langle y, (0.\dot{4} \wedge 0.5), (0.3 \vee 0.4), (0.5 \vee 0.6)\rangle\}$$

$$\dot{A} \sqcap \dot{C} = \{\langle \dot{x}, 0.4,0.5,0.6\rangle, \langle \dot{y}, 0.4,0.4,0.6\rangle\}$$

The complement in neutrosophic sense of $\dot{C}$ is

$$\dot{C}^{c} = \{\langle \dot{x}, 0.4,0.3,0.6\rangle, \langle \dot{y}, 0.5,0.4,0.6\rangle\}^{c}$$

$$\dot{C}^{c} = \{\langle x, \dot{0}.6, 1-0.3,0.4\rangle, \langle \dot{y}, 0.6,1-0.4,0.5\rangle\}$$

$\dot{C}^c = \{\langle x, \dot{0}.6, 0.7, 0.4\rangle, \langle \dot{y}, 0.6, 0.6, 0.5\rangle\}$

Theorem 1 Let $\dot{A}, \dot{B} \in \dot{N}(\dot{X})$. Then

- $\dot{A} \sqcap \dot{A} = A \text{ and } \dot{A} \sqcup \dot{A} = \dot{A}$
- $\dot{A} \sqcap \dot{B} = \dot{B} \sqcap \dot{A} \text{ and } \dot{B} \sqcup \dot{A} = \dot{A} \sqcup \dot{B}$
- $\dot{A} \sqcap \varphi = \varphi \text{ and } \dot{A} \sqcap \dot{X} = \dot{A}$
- $\dot{A} \sqcup \varphi = \varphi \text{ and } \dot{A} \sqcup \dot{X} = \dot{X}$
- $\dot{A} \sqcap (B \sqcap \dot{C}) = (\dot{A} \sqcap B) \dot{\sqcap} \dot{C} \text{ and } \dot{A} \sqcup (B \sqcup \dot{C}) = (\dot{A} \sqcup B) \dot{\sqcup} \dot{C}$
- $\left(\dot{A}^c\right)^c = \dot{A}$

Theorem 2 Let $\dot{A}, \dot{B} \in \dot{N}(\dot{X})$. Then

- $\left(\sqcap_{i\in I} \dot{A}_i\right)^c = \sqcap_{i\in I} \dot{A}_i{}^c$
- $\left(\sqcup_{i\in I} \dot{A}_i\right)^c = \sqcup_{i\in I} \dot{A}_i{}^c$

Theorem 3 Let $\dot{A}, \dot{B} \in \dot{N}(\dot{X})$. Then

- $\dot{B} \sqcap \left(\sqcup_{i\in I} \dot{A}_i\right) = \sqcup_{i\in I} (\dot{B} \sqcap \dot{A}_i)$
- $\dot{B} \sqcup \left(\sqcap_{i\in I} \dot{A}_i\right) = \sqcap_{i\in I} (\dot{B} \sqcup \dot{A}_i)$

3 Triangular neutrosophic topological spaces

Definition 3.1 Let $\dot{\tau} \subseteq \dot{N}(\dot{X})$, then $\dot{\tau}$ as neutrosophic topology on $\dot{X}$

- $\dot{X} \text{ and } \varphi \in \dot{\tau}$.
- The union and intersection of any number of neutrosophic sets in $\dot{\tau}$ belong to $\dot{\tau}$.

The pair $(\dot{X}, \dot{\tau})$ mentioned as neutrosophic topology space over $\dot{X}$.

Definition 3.2 If $(\dot{X}, \dot{\tau})$ be neutrosophic topological space over $\dot{X}$ then,

- $\dot{\phi} \text{ and } \dot{X} \text{ as neutrosophic closed sets over } \dot{X}$.

- The union and intersection of any two neutrosophic closed sets is a neutrosophic closed sets over $\dot{X}$.

Example 2: Let $\dot{X} = \{a.b\} and\ \dot{A} \in \dot{N}(\dot{X})$ so,

$A = \{\langle a, 0.4, \dot{0}.6, 0.\dot{8}\rangle, \langle b, 0.3, \dot{0}.5, 0.7\rangle\}$

Hence, $\dot{\tau} = \{\phi, \dot{X}, \dot{A}\}$ is neutrosophic topology on $\dot{X}$.

Example 3: Let $\dot{X} = \{a.b\} and\ \dot{A} \in \dot{N}(\dot{X})$ so,

$$\dot{A} = \{\langle \dot{x}, (0.2,0.4,0.6)(0.3,0.5,0.8)(0.2,0.8,0.8)\rangle, \langle y, (0.4,0.7,0.2)(0.6,0.7,0.2)(0.4,0.5,0.6)\rangle\}$$

$$\dot{B} = \{\langle \dot{x}, (0.8,02,0.3)(0.4,0.6,0.1)(0.7,0.3,0.5)\rangle, \langle \dot{y}, (0.1,0.2,0.4)(0.4,0.2,0.7)(0.1,0.2,0.4)\rangle\}$$

Apply formula $\frac{\acute{a}+\acute{b}+\acute{c}}{3}$,

$\dot{A} = \{\langle \dot{x}, 0.4,0.5,0.6\rangle, \langle \dot{y}, 0.4,0.3,0.5\rangle\}$

$\dot{B} = \{\langle \dot{x}, 0.4,0.3,0.6\rangle, \langle \dot{y}, 0.2,0.4,0.3\rangle\}$

Then, $\dot{\tau}_1 = \{\phi, \dot{X}, \dot{A}\}$ and $\dot{\tau}_2 = \{\phi, \dot{X}, \dot{A}\}$ are neutrosophic topology on $\dot{X}$. Here, $\dot{\tau}_1 \cup \dot{\tau}_1 = \{\phi, \dot{X}, \dot{A}, \dot{B}\}$ is not neutrosophic on $\dot{X}$. The reason is that: $\dot{A} \sqcap \dot{B} \notin \dot{\tau}_1 \cup \dot{\tau}_1$. Hence, it's not a neutrosophic topological space over $\dot{X}$.

Theorem 4: If $(\dot{X}, \dot{\tau})$ be neutrosophic topological space over $\dot{X}$ and $\dot{A}, \dot{B} \in \dot{N}(\dot{X})$ then:

i. $i\dot{n}t(\emptyset) = \dot{\phi}\ and\ int\ (\dot{X}) = (\dot{X})$

ii. $i\dot{n}t(A) \sqsubseteq \dot{A}$

iii. $\dot{A}$ *is neutrosophic open if and only if* $\dot{A} = int(\dot{A})$.

iv. $int(int(\dot{A})) = int(\dot{A})$.

v. $\dot{A} \sqsubseteq \dot{B}$ *implies* $i\dot{n}t(A) \sqsubseteq int(\dot{B})$.

vi. $i\dot{n}t(A) \sqcup int(\dot{B}) \sqsubseteq int(\dot{A} \sqcup \dot{B})$.

vii. $int(\dot{A} \sqcap \dot{B}) = int(\dot{A}) \sqcap int(\dot{B})$.

Proof: i. and ii. are obvious.

iii. $\dot{A}$ is neutrosophic open set over $\dot{X}$, as well as, $\dot{A}$ is itself a neutrosophic set over $\dot{X}$ which also contain $\dot{A}$. The largest neutrosophic open set contain in $\dot{A}$ is $\dot{A}$ and $int(\dot{A})$=$\dot{A}$. Conversely, int ($\dot{A}$)=$\dot{A}$ hence, $\dot{A} \in \dot{\tau}$.

iv. If $int(\dot{A})$=$\dot{B}. so, int(\dot{B}) = \dot{B}$ from above, $int\left(int(\dot{A})\right) = int(\dot{A})$.

v. As, $\dot{A} \sqsubseteq \dot{B}$. As $int(\dot{A}) \sqsubseteq \dot{A} \sqsubseteq \dot{B}.as\ int(\dot{A})$ is a neutrosophic subset of $\dot{B}$. So, $int(\dot{A}) \sqsubseteq int(\dot{B})$.

vi. It's clear $\dot{A} \sqsubseteq \dot{A} \sqcup \dot{B}\ and\ \dot{B} \sqsubseteq \dot{A} \sqcup \dot{B}$ thus, $int(\dot{A}) \sqsubseteq int(\dot{A} \sqcup \dot{B})\ and\ int(\dot{B}) \sqsubseteq int(\dot{A} \sqcup \dot{B})$ hence $int(\dot{A}) \sqcup int(\dot{B}) \sqsubseteq int(\dot{A} \sqcup \dot{B})$ by above.

Vii. If $(\dot{A} \sqcap \dot{B}) \sqsubseteq int(\dot{A})\ and\ (\dot{A} \sqcap \dot{B}) \sqsubseteq int(\dot{B})$ by above, so, $(\dot{A} \sqcap \dot{B}) \sqsubseteq int(\dot{A}) \sqcap int(\dot{B})$. Also, $int(\dot{A}) \sqsubseteq \dot{A}\ and\ int(\dot{B}) \sqsubseteq \dot{B}$ we have, $int(\dot{A}) \sqcap int(\dot{B}) \sqsubseteq \dot{A} \sqcap \dot{B}$. These make $(\dot{A} \sqcap \dot{B}) = int(\dot{A}) \sqcap int(\dot{B})$.

Example 4: Let $\dot{X} = \{\dot{x}, \dot{y}\}$ and $\dot{A}, \dot{B}, \dot{C} \in \dot{N}(\dot{x})$ then:

$$\dot{A} = \{\langle \dot{x}, (0.3,0.3,0.3)(0.3,0.3,0.3)(0.3,0.3,0.3)\rangle, \langle y, (0.5,0.5,0.5)(0.5,0.5,0.5)(0.5,0.5,0.5)\rangle\}$$

$$\dot{B} = \{\langle \dot{x}, (0.4,04,0.4)(0.4,0.4,0.4)(0.4,0.4,0.4)\rangle, \langle \dot{y}, (0.7,0.7,0.7)(0.7,0.7,0.7)(0.7,0.7,0.7)\rangle\}$$

$$\dot{C} = \{\langle \dot{x}, (0.2,0.2,0.2)(0.2,0.2,0.2)(0.2,0.2,0.2)\rangle, \langle \dot{y}, (0.6,0.6,0.6)(0.6,0.6,0.6)(0.6,0.6,0.6)\rangle\}$$

Apply formula $\frac{\acute{a}+\acute{b}+\acute{c}}{3}$,

$$\dot{A} = \{\langle \dot{x}, 0.3,0.3,0.3\rangle, \langle \dot{y}, 0.5,0.5,0.5\rangle\}$$

$\dot{B} = \{\langle \dot{x}, 0.4,0.4,0.4\rangle, \langle \dot{y}, 0.7,0.7,0.7\rangle\}$

$\dot{C} = \{\langle \dot{x}, 0.2,0.2,0.2\rangle, \langle \dot{y}, 0.6,0.6,0.6\rangle\}$

Then, $\dot{\tau} = \{\dot{\emptyset}, \dot{X}, \dot{A}\}$ is soft topological space over $\dot{X}$. $int(\dot{B}) = \dot{\phi}, int(\dot{C}) = \dot{\phi}$ and $(\dot{B} \sqcup \dot{C}) = \dot{A}$. Moreover, $int(\dot{B}) \sqcup int(\dot{C}) \neq int(\dot{B} \sqcup \dot{C})$.

Theorem 5: If $(\dot{X}, \dot{\tau})$ be neutrosophic topological space over $\dot{X}$ and $\dot{A}, \dot{B} \in \dot{N}(\dot{X})$ then:

1) $cl(\emptyset) = \emptyset \ and \ cl(\dot{X}) = \dot{X}$.
2) $\dot{A} \sqsubseteq cl(\dot{A})$.
3) $\dot{A}$ can be consider as neutrosophic closed set if and only if $\dot{A} = cl(\dot{A})$.
4) $cl\left(cl(\dot{A})\right) = cl(\dot{A})$.
5) $\dot{A} \sqsubseteq \dot{B}$ implies as $cl(\dot{A}) \sqsubseteq cl(\dot{B})$
6) $cl(\dot{A} \sqcup \dot{B}) = cl(\dot{A}) \sqcup cl(\dot{B})$.
7) $cl(\dot{A} \sqcap \dot{B}) \sqsubseteq cl(\dot{A}) \sqcap cl(\dot{B})$.

Proof: 1, 2, 6, and 7 are clear, as well as done previously above.

3) Suppose that $\dot{A}$ is neutrosophic closed set over $\dot{X}$, here $\dot{A}$ contain $\dot{A}$ and it is itself closed set over $\dot{X}$. $\dot{A}$ can be consider as smallest neutrosophic closed set contains $\dot{A}$ such as $A \dot{=} cl(\dot{A})$. Conversely, $A \dot{=} cl(\dot{A}) \ as \ \dot{A}$ is small one neutrosophic closed set over $\dot{X}$ contains $\dot{A}$.

4) by above case, $A \dot{=} cl(\dot{A})$, $\dot{A}$ is neutrosophic closed set.

5) $\dot{A} \sqsubseteq \dot{B}$. We can clearly see every neutrosophic closed super set of $\dot{B}$ is also neutrosophic closed super set of $\dot{A}$. Hence, $cl(\dot{A}) \sqsubseteq cl(\dot{B})$.

Example 5: Let $\dot{X} = \{\dot{x}, \dot{y}\}$ and $\dot{A}, \dot{B}, \dot{C} \in \dot{N}(\dot{x})$ then:

$$\dot{A} = \{\langle \dot{x}, (0.3,0.3,0.3)(0.3,0.3,0.3)(0.3,0.3,0.3)\rangle, \langle y, (0.5,0.5,0.5)(0.5,0.5,0.5)(0.5,0.5,0.5)\rangle\}$$

$$\dot{B} = \{\langle \dot{x}, (0.2,0.2,0.2)(0.2,0.2,0.2)(0.2,0.2,0.2)\rangle, \langle \dot{y}, (0.6,0.6,0.6)(0.6,0.6,0.6)(0.6,0.6,0.6)\rangle\}$$

Apply formula $\frac{\acute{a}+\acute{b}+\acute{c}}{3}$,

$$\dot{A} = \{\langle \dot{x}, 0.3,0.3,0.3\rangle, \langle \dot{y}, 0.5,0.5,0.5\rangle\}$$

$$\dot{B} = \{\langle \dot{x}, 0.2,0.2,0.2\rangle, \langle \dot{y}, 0.6,0.6,0.6\rangle\}$$

Then,

$$\dot{\tau} = \{\dot{\emptyset}, \dot{X}, \dot{A}, \dot{B}, \dot{A} \sqcap \dot{B}, \dot{A} \sqcup \dot{B}\}$$

After taking the compliment,

$$\{\dot{\emptyset}^c, \dot{X}^c, \dot{A}^c, \dot{B}^c, (\dot{A} \sqcap \dot{B})^c, (\dot{A} \sqcup \dot{B})^c\}$$

Therefore,

$$\dot{A}^c = \{\langle \dot{x}, 0.6,0.7,0.6\rangle, \langle \dot{y}, 0.5,0.5,0.5\rangle\}$$

$$\dot{B}^c = \{\langle \dot{x}, 0.2,0.8,0.2\rangle, \langle \dot{y}, 0.6,0.4,0.6\rangle\}$$

$$(\dot{A} \sqcap \dot{B})^c = \{\langle \dot{x}, 0.2,0.7,0.3\rangle\langle y, 0.6,\dot{0}.5,0.5\rangle\}$$

$$(\dot{A} \sqcup \dot{B})^c = \{\langle \dot{x}, 0.2,0.8,0.3\rangle, \langle y, 0.5,\dot{0}.5,0.6\rangle\}$$

$$\dot{A} \sqcap \dot{B} = \{\langle \dot{x}, 0.2,0.8,0.3\rangle, \langle y, 0.5,\dot{0}.5,0.6\rangle\}$$

$$cl(\dot{A}) = \dot{X}$$

$$cl(\dot{B}) = \dot{X}$$

$$cl(A \sqcap B) \dot{=} (\dot{A} \sqcup \dot{B})^c$$

$$cl(A \sqcap B) \sqsubseteq \dot{cl}(\dot{A}) \sqcap cl(\dot{B}).$$

Theorem 6: Let, $(\dot{X}, \dot{\tau})$ be neutrosophic topological space over $\dot{X}$ and $\dot{A}, \dot{B} \in \dot{N}(\dot{X})$ then:

i. $(\dot{fr}(\dot{A}))^c = ext(\dot{A}) \sqcup int(\dot{A})$.

ii. $cl(\dot{A}) = int(\dot{A}) \sqcup \dot{fr}(\dot{A})$.

Proof. $\dot{A}, \dot{B} \in \dot{N}(\dot{X})$. Then,

Here we have,

$(\dot{fr}(\dot{A}))^c = (cl(\dot{A}) \sqcap \dot{fr}(\dot{A}^c))^c$

$(\dot{fr}(\dot{A}))^c = (cl(\dot{A}))^c \sqcup (\dot{fr}(\dot{A}^c))^c$

$(\dot{fr}(\dot{A}))^c = (cl(\dot{A}))^c \sqcup (\dot{int}(\dot{A}^c))^c$

$ext(\dot{A}) \sqcup int(\dot{A})$

$int(\dot{A}) \sqcup \dot{fr}(\dot{A}) = int(\dot{A}) \sqcup (cl(\dot{A}) \sqcap \dot{fr}(\dot{A}^c))$

$int(\dot{A}) \sqcup \dot{fr}(\dot{A}) = int(\dot{A}) \sqcup (cl(\dot{A}) \sqcap (\dot{int}(\dot{A}) \sqcup fr(\dot{A}^c))$

$int(\dot{A}) \sqcup \dot{fr}(\dot{A}) = cl(\dot{A}) \sqcap (int(\dot{A}) \sqcup int(\dot{A}))^c$

$int(\dot{A}) \sqcup \dot{fr}(\dot{A}) = cl(\dot{A}) \sqcap \dot{X}$

$int(\dot{A}) \sqcup \dot{fr}(\dot{A}) = cl(\dot{A})$.

Theorem 7: Let, $(\dot{X}, \dot{\tau})$ be neutrosophic topological space over $\dot{X}$ and $\dot{A}, \dot{B} \in \dot{N}(\dot{X})$ then:

i. $\dot{fr}(\dot{A}) \sqcap int(\dot{A}) = \emptyset$

ii. $\dot{fr}(int(\dot{A})) \sqsubseteq \dot{fr}(\dot{A})$

Proof: $\dot{A} \in \dot{N}(\dot{X})$. then,

i. is clear.
To prove ii. Let,

$\dot{fr}(int(\dot{A})) = cl(int(\dot{A})) \sqcap cl(int(\dot{A}))$

$\dot{fr}(int(\dot{A})) = cl(int(\dot{A})) \sqcap \dot{fr}(\dot{A}^c)$

$\dot{fr}(int(\dot{A})) = cl(\dot{A}) \sqcap \dot{fr}(\dot{A}^c)$

$\dot{f}r(int(\dot{A})) \sqsubseteq \dot{f}r(\dot{A})$

Definition: Let, $(\dot{X}, \dot{\tau})$ be neutrosophic topological space and $\dot{Y}$ is non empty subset of $\dot{X}$. Neutrosophic relative topology as:

$\dot{\tau}_Y = \{\dot{A} \sqcap \dot{Y} : \dot{A} \in \dot{\tau}\}$

$$Y(\dot{x}) = \begin{cases} \langle 1,0,0 \rangle & \dot{x} \in \dot{Y} \\ \langle 0,1,1 \rangle & otherwise \end{cases}$$

Hence, $(\dot{X}, \dot{\tau}_Y)$ as neutrosophic subspace of $(\dot{X}, \dot{\tau})$.

Example 6: Let $\dot{X} = \{\dot{a}, \dot{b}, \dot{c}\}$, $\dot{Y} = \{\dot{a}, \dot{b}\} \subseteq \dot{X}$ and $\dot{A}, \dot{B} \in \dot{N}(\dot{X})$ then,

$\dot{A}$
$= \{\langle \dot{x}, (0.3,0.5,0.7)(0.2,0.6,0.1)(0.4,0.6,0.7) \rangle, \langle y, (0.2,0.5,0.9)(0.3,0.1,0.8)(0.5,0.2,0.3) \rangle\}$

$\dot{B}$
$= \{\langle \dot{x}, (0.2,0.6,0.4)(0.6,0.2,0.7)(0.2,0.1,0.6) \rangle, \langle \dot{y}, (0.6,0.8,0.7)(0.7,0.6,0.3)(0.1,0.5,0.6) \rangle\}$

Apply formula $\frac{\acute{a}+\acute{b}+\acute{c}}{3}$,

$\dot{A} = \{\langle \dot{x}, 0.5,0.3,0.6 \rangle, \langle \dot{y}, 0.5,0.4,0.3 \rangle\}$

$\dot{B} = \{\langle \dot{x}, 0.4,0.5,0.3 \rangle, \langle \dot{y}, 0.7,0.5,0.4 \rangle\}$

Thus,

$\dot{\tau} = \{\dot{\emptyset}, \dot{X}, \dot{A}, \dot{B}, \dot{A} \sqcap \dot{B}, \dot{A} \sqcup \dot{B}\}$

As neutrosophic topology on $\dot{X}$. As well as,

$\dot{\tau}_Y = \{\dot{\emptyset}, \dot{Y}, \dot{C}, \dot{M}, \dot{L}, \dot{K}\}$ such that, $\dot{C} = \dot{Y} \sqcap \dot{A}$. $\dot{M} = \dot{Y} \sqcap \dot{B}$, $\dot{L} = \dot{Y} \sqcap (\dot{A} \sqcap \dot{B})$ and $\dot{K} = \dot{Y} \sqcap (\dot{A} \sqcup \dot{B})$.

Conclusion

In this current article, we rearrange the neutrosophic set operations and design triangular neutrosophic topology on the structure of neutrosophic topology with defuzzification. We introduce some properties linked to operations. We also clear the neutrosophic topology structure of neutrosophic sets. Moreover, we believe, with these new approaches, the researcher will able to enhance new possibilities in neutrosophic topology.

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Neutrosophic Random Variables

Mohamed Bisher Zeina [1], Ahmed Hatip[2]

[1] Department of Mathematical Statistics, University of Aleppo, Aleppo, Syria; bisher.zeina@gmail.com.
[2] Department of Mathematics, University of Gaziantep, Turkey; kollnaar5@gmail.com

* Correspondence: bisher.zeina@gmail.com Tel.: (+963969602951)

Abstract: In this paper, general definition of neutrosophic random variables is introduced and its properties are presented. Concepts of probability distribution function, cumulative distribution function, expected value, variance, standard deviation, mean deviation, r^th quartiles, moments generating function and characteristic function in crisp logic are generalized to neutrosophic logic. Many solved problems and applications are presented which show the power of the study and show the ability of applying the results in various domains including quality control, stochastic modeling, reliability theory, queueing theory, decision making, electrical engineering, … etc.

Keywords: Expected Value; Variance; Standard Deviation; Probability Density Function; Cumulative Distribution Function; Moments Generating Function; Characteristic Function; Neutrosophic Logic.

1. Introduction

Neutrosophic logic is an extension of intuitionistic fuzzy logic by adding indeterminacy component (I) where $I^2 = I, \dots, I^n = I, 0 \cdot I = 0$; $n \in N$ and I^{-1} is undefined [1], [2]. Neutrosophic logic has wide applications in many fields including decision making [3], [4], [5], machine learning [6], [7], intelligent disease diagnosis [8], [9], communication services [10], pattern recognition [11], social network analysis and e-learning systems [12], physics [13], [14], … etc.

In probability theory, F. Smarandache defined the neutrosophic probability measure as a mapping $NP: X \to [0,1]^3$ where X is a neutrosophic sample space, and defined the probability function to take the form $NP(A) = \big(ch(A), ch(neutA), ch(antiA)\big) = (\alpha, \beta, \gamma)$ where $0 \le \alpha, \beta, \gamma \le 1$ and $0 \le \alpha + \beta + \gamma \le 3$ [15], also researchers introduced many neutrosophic probability distributions like Poisson, exponential, binomial, normal, uniform, Weibull, …etc. [2], [16], [17], [18]. Researchers also presented the concept of neutrosophic queueing theory in [19], [20] that is one branch of neutrosophic stochastic modelling. Researchers also studied neutrosophic time series prediction and modelling in many cases like neutrosophic moving averages, neutrosophic logarithmic models, neutrosophic linear models, … etc. [21], [22], [23].

Neutrosophic logic has solved many decision-making problems efficiently like evaluating green credit rating, personnel selection, … etc. [24], [25], [26], [27].

In this paper we will suggest a generalization to classical random variable to deal with imprecise, uncertainty, ambiguity, vagueness, enigmatic adding the indeterminacy part to its form, then we will find several characteristics of this neutrosophic random variable including expected value, variance, standard deviation, moments generating function and characteristic function and study its properties.

This extension lets us build and study many stochastic models in the future that help us in modelling, simulation, decision making, prediction and classification specially in the cases of incomplete data and indeterminacy.

2. Terminologies

We present here some basic definitions and axioms of neutrosophic logic and neutrosophic probability.

2.1 Some definitions

Definition 1 [28]: Let X be a non-empty fixed set. A neutrosophic set A is an object having the form $\{x,(\mu A(x),\delta A(x),\gamma A(x)) : x \in X\}$, where $\mu A(x)$, $\delta A(x)$ and $\gamma A(x)$ represent the degree of membership, the degree of indeterminacy , and the degree of non-membership respectively of each element $x \in X$ to the set A .

Definition 2 [29]: Let K be a field, the neutrosophic file generated by $\langle K \cup I \rangle$ which is denoted by $K(I) = \langle K \cup I \rangle$.

Definition 3 [2]: Classical neutrosophic number has the form $a + bI$ where a,b are real or complex numbers and I is the indeterminacy such that $0 \cdot I = 0$ and $I^2 = I$ which results that $I^n = I$ for all positive integers n.

Definition 4 [15]: The neutrosophic probability of event A occurrence is $NP(A) = \big(ch(A), ch(neutA), ch(antiA)\big) = (T, I, F)$ where T, I, F are standard or nonstandard subsets of the nonstandard unitary interval $]^{-}0,1^{+}[$.

Among this paper, we will denote probability density function by PDF, probability mass function by PMF, cumulative distribution function by CDF, moments generating function by MGF, characteristic function by CF.

3. Neutrosophic Random Variables

In *[15]* Smarandache defined the neutrosophic random variable that it is a variable that may have and indeterminate outcome, in the following definition we are going to represent this indeterminacy by mathematical formula, then we are going to find the properties of neutrosophic random variable.

Definition 3.1: Neutrosophic Random Variable

Consider the real valued crisp random variable X which is defined as follows:

$$X : \Omega \to R$$

Where Ω is the events space. We define neutrosophic random variable X_N as the following:

$$X_N : \Omega \to R(I)$$

And:

$$X_N = X + I$$

Where I is indeterminacy.

3.1: PDF and CDF of neutrosophic random variables

Consider the neutrosophic random $X_N = X + I$, Where CDF of X is $F_X(x) = P(X \le x)$ then:

$$F_{X_N}(x) = F_X(x - I) \quad (1)$$

$$f_{X_N}(x) = f_X(x - I) \quad (2)$$

Proof:

$$F_{X_N}(x) = P(X_N \le x) = P(X + I \le x) = P(X \le x - I) = F_X(x - I)$$

By taking the derivative according to x we get:

$$f_{X_N}(x) = \frac{dF_{X_N}(x)}{dx} = \frac{dF_X(x - I)}{dx} \cdot \frac{d(x - I)}{dx} = f_X(x - I)$$

3.2: Expected value of neutrosophic random variable

Consider the neutrosophic random variable $X_N = X + I$, we can find its expected value as follows:

$$E(X_N) = E(X) + I \quad (3)$$

Proof:

If X is continuous then:

$$E(X_N) = E(X+I) = \int_x (x+I)f(x)dx = \int_x xf(x)dx + I\int_x f(x)dx = E(X) + I$$

Where $\int_x f(x)dx = 1$ because it is a pdf

If X is discrete then:

$$E(X_N) = E(X+I) = \sum_x (x+I)f(x) = \sum_x xf(x) + I\sum_x f(x) = E(X) + I$$

Properties of expected value of a neutrosophic random variable

1. $E(aX_N + b + cI) = aE(X_N) + b + cI\,; a, b, c \in R$

   Proof: straight forward.

2. If X_N, Y_N are two neutrosophic random variables, then $E(X_N \pm Y_N) = E(X_N) \pm E(Y_N)$

   Proof: straight forward.

3. $E[(a+bI)X_N] = E(aX_N + bIX_N) = E(aX_N) + E(bIX_N) = aE(X_N) + bIE(X_N)\,; a, b \in R$

   Proof: straight forward.

4. $|E(X_N)| \le E|X_N|$

   Proof:

   If X is continuous:

$$|E(X_N)| = \left|\int_x (x+I)f(x)dx\right| \le \int_x |(x+I)|f(x)dx = E|X_N|$$

   Where $|f(x)| = f(x)$ because it is a PDF

   If X is discrete:

$$|E(X_N)| = \left|\sum_x (x+I)f(x)\right| \le \sum_x |(x+I)|f(x) = E|X_N|$$

3.3: Variance of neutrosophic random variable

Consider the neutrosophic random variable $X_N = X + I$, we can prove that its variance is equal to X's variance, i.e.:

$$V(X_N) = V(X) \tag{4}$$

Proof:

Whatever is X_N, discrete or continuous we can write:

$$V(X_N) = E[X_N - E(X_N)]^2 = E[X + I - E(X) - I]^2 = E[X - E(X)]^2 = V(X)$$

Example 3.1

Let X be a random variable with probability density function given as follows:

$$f_X(x) = 2x\,; 0 \le x \le 1$$

(a) We will find PDF of $X_N = X + I$ then proof that it's a density function (it's integral equals to one)

(b) We will calculate the expected value of X_N.

(c) We will calculate the variance of X_N.

Solution:

(a)

Using equation (1):

$$f_{X_N}(x) = f_X(x - I) = 2(x - I); 0 \le x - I \le 1$$

$$f_{X_N}(x) = 2x - 2I\;;\; I \le x \le 1 + I$$

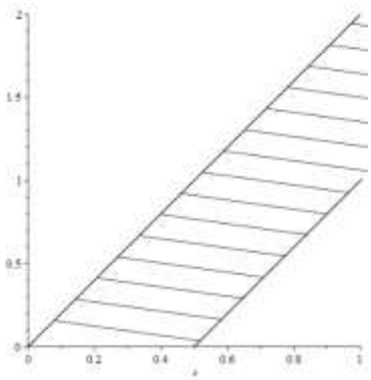

Fig (1)

Let's prove that $f_{X_N}(x)$ is a density function

$$\int_{I}^{1+I}(2x-2I)dx=[x^2-2Ix]_I^{1+I}=(1+I)^2-2I(1+I)-I^2+2I^2=1+2I+I^2-2I-2I^2-I^2+2I^2$$

$$=1+2I+I-2I-2I-I+2I=1$$

(b) using equations (3), (4):

$$E(X_N)=E(X)+I=\int_0^1 2x^2dx+I=\frac{2}{3}+I$$

(c)

$$V(X_N)=V(X)=\int_0^1\left(x-\frac{2}{3}\right)^2 2xdx=\frac{1}{18}$$

3.4: Mean deviation of neutrosophic random variable:

The mean deviation of neutrosophic random variable denoted by $M.D(X_N)$ is:

$$\boldsymbol{M.D(X_N)=M.D(X)=E|X-E(X)|} \quad (5)$$

Proof:

$$M.D(X_N)=E|X_N-E(X_N)|=E|X+I-E(X+I)|=E|X+I-E(X)-I|=M.D(X)$$

3.5: The r^th^ quartile of neutrosophic continuous random variable:

The r^th^ quartile of neutrosophic random variable denoted by Q_N^r is:

$$\boldsymbol{\int_{-\infty}^{Q_N^r} f_{X_N}(x)dx=\frac{r}{4};r=1,2,3} \quad (6)$$

We call Q_N^1, Q_N^2 and Q_N^3 the neutrosophic first, second and third quartiles respectively.

Example 3.2

Let X_N be the neutrosophic random variable defined in example 3.1, let's calculate it's 3 quartiles.

Solution:

We have

$$f_{X_N}(x)=2x-2I\ ;\ I\le x\le 1+I$$

So, using equation (6):

$$\int_I^{Q_N^r}(2x-2I)dx=\frac{r}{4};r=1,2,3$$

For $r=1$ we get:

$$\int_{I}^{Q_N^1} (2x - 2I)dx = \frac{1}{4}$$

$$[x^2 - 2Ix]_I^{Q_N^1} = \frac{1}{4}$$

$$Q_N^{1\,2} - 2IQ_N^1 - I^2 + 2I^2 = \frac{1}{4}$$

$$Q_N^{1\,2} - 2IQ_N^1 + I = \frac{1}{4}$$

Solving the neutrosophic equation respect to Q_N^1 we get:

$$\Delta = b^2 - 4ac = 4I - 4\left(I - \frac{1}{4}\right) = 4I - 4I + 1 = 1$$

$$(Q_N^1)_1 = \frac{-b - \sqrt{\Delta}}{2a} = \frac{2I - 1}{2} = -\frac{1}{2} + I$$

Rejected because $I \le x \le 1 + I$.

$$(Q_N^1)_2 = \frac{-b + \sqrt{\Delta}}{2a} = \frac{2I + 1}{2} = \frac{1}{2} + I$$

Accepted.

For $r = 2$ we get:

$$Q_N^{2\,2} - 2IQ_N^2 + I = \frac{2}{4} = \frac{1}{2}$$

Solving the neutrosophic equation respect to Q_N^2 we get:

$$\Delta = b^2 - 4ac = 4I - 4\left(I - \frac{1}{2}\right) = 4I - 4I + 2 = 2$$

$$(Q_N^2)_1 = \frac{-b - \sqrt{\Delta}}{2a} = \frac{2I - \sqrt{2}}{2} = -\frac{\sqrt{2}}{2} + I$$

Rejected.

$$(Q_N^2)_2 = \frac{-b + \sqrt{\Delta}}{2a} = \frac{2I + \sqrt{2}}{2} = \frac{\sqrt{2}}{2} + I$$

Accepted.

For $r = 3$ we get:

$$Q_N^{3\,2} - 2IQ_N^3 + I = \frac{3}{4}$$

Solving the neutrosophic equation respect to Q_N^3 we get:

$$\Delta = b^2 - 4ac = 4I - 4\left(I - \frac{3}{2}\right) = 4I - 4I + 6 = 6$$

$$(Q_N^3)_1 = \frac{-b - \sqrt{\Delta}}{2a} = \frac{2I - \sqrt{6}}{2} = -\frac{\sqrt{6}}{2} + I$$

Rejected.

$$(Q_N^3)_2 = \frac{-b + \sqrt{\Delta}}{2a} = \frac{2I + \sqrt{6}}{2} = \frac{\sqrt{6}}{2} + I$$

Accepted.

3.6: MGF of neutrosophic random variable

Consider the neutrosophic random $X_N = X + I$ then its MGF will be:

$$M_{X_N}(t) = e^{tI} M_X(t) \tag{7}$$

Proof:

$$M_{X_N}(t) = E(e^{tX_N}) = E\left(e^{t(X+I)}\right) = E(e^{tX}e^{tI}) = e^{tI}E(e^{tX}) = e^{tI}M_X(t)$$

Properties:

1. $M_{X_N}(0) = 1$

Proof: Straight forward.

2. $\frac{dM_{X_N}(0)}{dt} = E(X_N)$

Proof:

$$\frac{dM_{X_N}(t)}{dt}|_{t=0}=\frac{de^{tI}M_X(t)}{dt}|_{t=0}=\frac{de^{tI}}{dt}M_X(t)|_{t=0}+\frac{dM_X(t)}{dt}e^{tI}|_{t=0}=Ie^{tI}M_X(t)|_{t=0}+M'_X(t)e^{tI}|_{t=0}$$
$$=IM_X(0)+M'_X(0)=I+E(X)=E(X_N)$$

3. $\frac{d^nM_{X_N}(t)}{dt^n}|_{t=0}=E(X_N^n)$

Proof: Straight forward.

4. If $Y_N=(a+bI)X_N+c+dI$ then $M_{Y_N}(t)=e^{(c+dI)t}e^{t(a+b)I}M_X\big((a+bI)t\big)$

Proof:

$$M_{Y_N}(t)=E(e^{tY_N})=E\left(e^{t[(a+bI)X_N+c+dI]}\right)=E\left(e^{t(a+bI)(X+I)}\,e^{(c+dI)t}\right)=e^{(c+dI)t}E\left(e^{t(a+bI)(X+I)}\right)$$
$$=e^{(c+dI)t}E\left(e^{t(aX+aI+bIX+bI^2)}\right)=e^{(c+dI)t}e^{t(a+b)I}E\left(e^{t(a+bI)X}\right)$$
$$=e^{(c+dI)t}e^{t(a+b)I}M_X\big((a+bI)t\big)$$

Theorem 3.5 CF of Neutrosophic Random Variable

Consider the neutrosophic random $X_N=X+I$ then its CF will be:

$$\varphi_{X_N}(t)=e^{itI}\varphi_X(t)\ ;i=\sqrt{-1} \qquad (8)$$

Proof:

$$\varphi_{X_N}(t)=E(e^{itX_N})=E\left(e^{it(X+I)}\right)=E(e^{itX}e^{itI})=e^{itI}E(e^{itX})=e^{itI}\varphi_X(t)$$

Properties:

1. $\varphi_{X_N}(0)=1$

Proof: Straight forward.

2. $|\varphi_{X_N}(t)|\le 1$, which means that CF always exists.

Proof:

$$|\varphi_{X_N}(t)|=|E(e^{itX_N})|\le E|e^{itX_N}|=E|\cos tX_N+\sin tX_N|=E|1|=1$$

3. $\frac{d\varphi_{X_N}(t)}{dt}|_{t=0}=iE(X_N)$

Proof:

$$\frac{d\varphi_{X_N}(t)}{dt}|_{t=0}=\frac{de^{itI}\varphi_X(t)}{dt}|_{t=0}=\frac{de^{itI}}{dt}\varphi_X(t)|_{t=0}+\frac{d\varphi_X(t)}{dt}e^{itI}|_{t=0}$$
$$=iIe^{itI}\varphi_X(t)|_{t=0}+\varphi'_X(t)e^{itI}|_{t=0}=iI\varphi_X(0)+\varphi'_X(0)=iI+iE(X)=i\big(I+E(X)\big)$$
$$=iE(X_N)$$

4. $\frac{d^n\varphi_{X_N}(t)}{dt^n}|_{t=0}=i^nE(X_N^n)$

Proof: Straight forward.

5. $\varphi_{X_N}(t)=M_{X_N}(it)$

Proof: Straight forward.

Example 3.3

Let X_N be the neutrosophic random variable defined in example 3.1 and let's find:

(a) $M_{X_N}(t)$.

(b) $E(X_N)$ Depending on properties of $M_{X_N}(t)$

(c) Conclude $\varphi_{X_N}(t)$ formula.

Solution

(a) Using equation (7):

$$M_{X_N}(t)=e^{tI}M_X(t)$$

But:

$$M_X(t)=\int_0^1 e^{tx}2xdx=\frac{2(te^t-e^t+1)}{t^2}$$

So:

$$M_{X_N}(t)=e^{tI}\frac{2(te^t-e^t+1)}{t^2}=2\frac{te^{t(1+I)}-e^{t(1+I)}+e^{tI}}{t^2}$$

(b) Using proved properties of $M_{X_N}(t)$ we get:

$$M'_{X_N}(t) = 2\frac{t^2\left(e^{t(1+I)} + (1+I)e^{t(1+I)}t - (1+I)e^{t(1+I)} + Ie^{tI}\right) - 2t\left(te^{t(1+I)} - e^{t(1+I)} + e^{tI}\right)}{t^4}$$
$$= 2\frac{t\left(e^{t(1+I)} + (1+I)e^{t(1+I)}t - (1+I)e^{t(1+I)} + Ie^{tI}\right) - 2\left(te^{t(1+I)} - e^{t(1+I)} + e^{tI}\right)}{t^3}$$
$$= 2\frac{te^{t(1+I)} + (1+I)e^{t(1+I)}t^2 - (1+I)te^{t(1+I)} + Ite^{tI} - 2te^{t(1+I)} + 2e^{t(1+I)} - 2e^{tI}}{t^3}$$

$$M'_{X_N}(0) = \frac{2}{3} + I = E(X_N)$$

(c) Using the proved property that $\varphi_{X_N}(t) = M_{X_N}(it)$ we get:

$$\varphi_{X_N}(t) = M_{X_N}(it) = 2\frac{ite^{it(1+I)} - e^{it(1+I)} + e^{itI}}{-t^2}$$

4. Applications and Future Research Directions

The results that are presented in this paper can be applied to define several concepts in neutrosophic probability theory that are not defined and not studied yet including stochastic processes, reliability theory models, quality control techniques, …etc. where all depend on the concept of neutrosophic random variables and it's properties. Also, these results can be applied in stochastic modelling and random numbers generating which is very important in simulation of probabilistic models.

We are looking forward to study the properties of neutrosophic probability distributions like Pareto, Gaussian, Gamma, Beta, … etc. when the distribution of random variables changes to $X_N = X + I$ i.e., when the random variable contains an indeterminant part so we can model and simulate many stochastic problems including arrivals and departures to services stations, lifetimes of units in manufacturing systems, loss models, …etc.

5. Conclusions

In this research, we firstly obtained a general definition of neutrosophic random variables, concepts of probability distribution function and cumulative distribution function. We focused on the neutrosophic representation and proved some properties. In addition, we showed the ability of applying the results in various domains including quality control, stochastic modeling, reliability theory, queueing theory, electrical engineering, …etc.

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Application of Neutrosophic Vague Nano Topological Spaces

Mary Margaret A[1*], Trinita Pricilla M[2]

[1]Research Scholar, Department of Mathematics, Nirmala College for Women, Coimbatore-641 018, India;

E-mail: marymargaret.a11@gmail.com

[2]Assistant Professor, Department of Mathematics, Nirmala College for Women, Coimbatore-641 018, India.

E-mail: abishai_kennet@yahoo.in

* Correspondence: marymargaret.a11@gmail.com

Abstract: This paper intends to presents a new structure of nano topological space by using neutrosophic vague sets. Here we define the neutrosophic vague nano topological spaces, various properties and characterizations related to these sets are also examined. Because of the increased volume of information available to physicians from advanced medical technology, the obtained information of each symptom with respect to a disease may contain truth, falsity and indeterminacy information, by using this information an application is also discussed with neutrosophic vague nano topological space. In this application we identified the risk factors for the cause of stroke attack through the concept of neutrosophic vague nano topology.

Keywords: Neutrosophic vague nano topological space, neutrosophic vague lower approximation, neutrosophic vague upper approximation and neutrosophic vague boundary.

1. Introduction

Recently, several theories have been proposed to deal with uncertainty, imprecision and vagueness. The concept of fuzzy sets was introduced in 1965 by Zadeh[14]. Using this fuzzy sets in 1986 intuitionistic fuzzy sets was introduced by Atanassov[5]. The theory of vague sets was first proposed and developed as an extension of fuzzy set theory by Gau and Buehrer[6]. Then, Smarandache[13], introduces the neutrosophic elements T, I, F which represent the membership, indeterminacy, and non-membership values respectively, where $]^{-}0,1^{+}[$ is that the non-standard unit interval in 1998. Shawkat Alkhazaleh[12], in 2015 introduced and constructed the concept of neutrosophic vague set. The concept of nano topological space was introduced by Lellis Thivagar[8]. Intuitionistic fuzzy nano topological space was introduced by Ramachandran[11] in 2017.

Medical diagnosis is process of investigation of a person's symptoms on the basis of diseases. From modern medical technology, a large amount of information available to medical experts due to whom medical diagnosis contains uncertain, inconsistent, indeterminate information and this information are mandatory in medical diagnosis. A characterized relationship between a symptom and a disease is usually based on these uncertain, inconsistent information which leads to us for decision making in a medical diagnosis. Mostly diagnosis problems have pattern recognition on the basis of which medical experts make their decision. Medical diagnosis has successful practical applications in several areas such as telemedicine, space medicine and rescue services etc. where access of human means of diagnosis is a difficult task. Thus, starting from the early time of Artificial Intelligence, medical diagnosis has got full attention from both computer science and computer applicable mathematics research society. Abdel-Basset,et.al.,[1,2,3,4] formulated various multi-criteria decision-making approach.

In this paper we use the neutrosophic vague sets in nano topological space and defined the neutrosophic vague nano topological space and discussed some of its properties. Here we also use the neutrosophic vague nano topological space in real time application in multi decision problem such as medical diagnosis to identify the vital factors for the Stroke in patients.

2. Preliminaries:

Definition 2.1:[10] Let U^* be a non-empty finite set of objects called the universe and R^* be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U^*, R^*) is said to be the approximation space. Let $X^* \subseteq U^*$.

i) The lower approximation of X^* with respect to R^* is denoted by $L_{R^*}(X^*)$. That is, $L_{R^*}(X^*) = \bigcup_{x \in U^*}\{R^*(x): R^*(x) \subseteq X^*\}$, where denotes the equivalence class determined by x.

ii) The upper approximation of X^* with respect to R^* denoted by $U_{R^*}(X^*)$. That is, $U_{R^*}(X^*) = \bigcup_{x \in U^*}\{R^*(x): R^*(x) \cap X^* \neq \emptyset\}$.

iii) The boundary region of X^* with respect to R^* is denoted by $B_{R^*}(X^*)$. That is, $B_{R^*}(X^*) = U_{R^*}(X^*) - L_{R^*}(X^*)$.

Definition 2.2:[8] Let U^* be an universe, R^* be an equivalence relation on U^* and $\tau_{R^*}(X^*) = \{U^*, \emptyset, L_{R^*}(X^*), U_{R^*}(X^*), B_{R^*}(X^*)\}$ where $X^* \subseteq U^*$. $\tau_{R^*}(X^*)$ satisfies the following axioms:

i) X^* and $\emptyset \in \tau_{R^*}(X^*)$.

ii) The union of the elements of any sub-collection of $\tau_{R^*}(X^*)$ is in $\tau_{R^*}(X^*)$.

iii) The intersection of the elements of any finite sub-collection of $\tau_{R^*}(X^*)$ is in $\tau_{R^*}(X^*)$.

Definition 2.3:[12] A neutrosophic vague set A_{NV}^{*} (NVS in short) on the universe of discourse X^{*} written as $A_{NV}^{*} = \left\{ \left\langle x; \hat{T}_{A_{NV}^{*}}(x); \hat{I}_{A_{NV}^{*}}(x); \hat{F}_{A_{NV}^{*}}(x) \right\rangle; x \in X^{*} \right\}$, whose truth membership, indeterminacy membership and false membership functions is defined as:

$$\hat{T}_{A_{NV}^{*}}(x) = \left[T^{-}, T^{+}\right], \hat{I}_{A_{NV}^{*}}(x) = \left[I^{-}, I^{+}\right], \hat{F}_{A_{NV}^{*}}(x) = \left[F^{-}, F^{+}\right]$$

where,

i) $T^{+} = 1 - F^{-}$

ii) $F^{+} = 1 - T^{-}$ and

iii) $^{-}0 \leq T^{-} + I^{-} + F^{-} \leq 2^{+}$.

Definition 2.4:[12] Let A_{NV}^{*} and B_{NV}^{*} be two NVSs of the universe U. If $\forall u_i \in U, \hat{T}_{A_{NV}^{*}}(u_i) \leq \hat{T}_{B_{NV}^{*}}(u_i);\ \hat{I}_{A_{NV}^{*}}(u_i) \geq \hat{I}_{B_{NV}^{*}}(u_i);\ \hat{F}_{A_{NV}^{*}}(u_i) \geq \hat{F}_{B_{NV}^{*}}(u_i)$, then the NVS A_{NV}^{*} is included by B_{NV}^{*}, denoted by $A_{NV}^{*} \subseteq B_{NV}^{*}$, where $1 \leq i \leq n$.

Definition 2.5:[12] The complement of NVS A_{NV}^{*} is denoted by $\left(A_{NV}^{*}\right)^{c}$ and is defined by

$$\hat{T}^{c}_{A_{NV}^{*}}(x) = \left[1 - T^{+}, 1 - T^{-}\right], \hat{I}^{c}_{A_{NV}^{*}}(x) = \left[1 - I^{+}, 1 - I^{-}\right], \hat{F}^{c}_{A_{NV}^{*}}(x) = \left[1 - F^{+}, 1 - F^{-}\right].$$

3. Neutrosophic Vague Nano Topological Spaces:

Definition 3.1: Let Ω be a non-empty set and $\mathcal{R}^{*}$ be an equivalence relation on Ω. Let Q be a neutrosophic vague set in Ω with the truth membership function $\hat{T}_Q$, the indeterminacy function $\hat{I}_Q$ and the false membership function $\hat{F}_Q$. The neutrosophic vague lower approximation, neutrosophic vague upper approximation and neutrosophic vague boundary of Q in the neutrosophic vague approximation space (Ω, $\mathcal{R}^{*}$) is denoted by NVLA(Q), NVUA(Q) and NVB(Q) are respectively defined as follows.

i) $NVLA(Q) = \{\langle x; \hat{T}_{LQ}(x); \hat{I}_{LQ}(x); \hat{F}_{LQ}(x)\rangle; y \in [x]_{\mathcal{R}^{*}}, x \in \Omega\}$.

ii) $NVUA(Q) = \{\langle x; \hat{T}_{UQ}(x); \hat{I}_{UQ}(x); \hat{F}_{UQ}(x)\rangle; y \in [x]_{\mathcal{R}^{*}}, x \in \Omega\}$.

iii) $NVB(Q) = NVUA(Q) - NVLA(Q)$.

Where $\hat{T}_{LQ}(x) = \wedge_{y \in [x]_{\mathcal{R}^{*}}} \hat{T}_Q(y)$; $\hat{I}_{LQ}(x) = \vee_{y \in [x]_{\mathcal{R}^{*}}} \hat{I}_Q(y)$; $\hat{F}_{LQ}(x) = \vee_{y \in [x]_{\mathcal{R}^{*}}} \hat{F}_Q(y)$

for neutrosophic vague set Q we have $\hat{T}_Q(y) = [T^-, T^+]$; $\hat{I}_Q(y) = [I^-, I^+]$; $\hat{F}_Q(y) = [F^-, F^+]$

so, $\hat{T}_{LQ}(x) = \left[\wedge_{y\in[x]_{\mathcal{R}^*}} T^-{}_Q(y), \wedge_{y\in[x]_{\mathcal{R}^*}} T^+{}_Q(y)\right]$;

$\hat{I}_{LQ}(x) = \left[\vee_{y\in[x]_{\mathcal{R}^*}} I^-{}_Q(y), \vee_{y\in[x]_{\mathcal{R}^*}} I^+{}_Q(y)\right]$ and

$\hat{F}_{LQ}(x) = \left[\vee_{y\in[x]_{\mathcal{R}^*}} F^-{}_Q(y), \vee_{y\in[x]_{\mathcal{R}^*}} F^+{}_Q(y)\right]$

And $\hat{T}_{UQ}(x) = \vee_{y\in[x]_{\mathcal{R}^*}} \hat{T}_Q(y)$; $\hat{I}_{UQ}(x) = \wedge_{\in[x]_{\mathcal{R}^*}} \hat{I}_Q(y)$; $\hat{F}_{UQ}(x) = \wedge_{y\in[x]_{\mathcal{R}^*}} \hat{F}_Q(y)$

for neutrosophic vague set Q we have $\hat{T}_Q(y) = [T^-, T^+]$; $\hat{I}_Q(y) = [I^-, I^+]$; $\hat{F}_Q(y) = [F^-, F^+]$

so, $\hat{T}_{UQ}(x) = \left[\vee_{y\in[x]_{\mathcal{R}^*}} T^-{}_Q(y), \vee_{y\in[x]_{\mathcal{R}^*}} T^+{}_Q(y)\right]$;

$\hat{I}_{UQ}(x) = \left[\wedge_{y\in[x]_{\mathcal{R}^*}} I^-{}_Q(y), \wedge_{y\in[x]_{\mathcal{R}^*}} I^+{}_Q(y)\right]$ and

$\hat{F}_{UQ}(x) = \left[\wedge_{y\in[x]_{\mathcal{R}^*}} F^-{}_Q(y), \wedge_{y\in[x]_{\mathcal{R}^*}} F^+{}_Q(y)\right]$

Definition 3.2: If (Ω, $\mathcal{R}^*$) is a neutrosophic vague approximation space and let $P, Q \subseteq \Omega$, then the following statements holds:

i) $NVLA(P) \subseteq P \subseteq NVUA(P)$.

ii) $NVLA(0_{NV}) = NVUA(0_{NV}) = 0_{NV}$ and $NVLA(1_{NV}) = NVUA(1_{NV}) = 1_{NV}$.

iii) $NVLA(P \cup Q) \supseteq NVLA(P) \cup NVLA(Q)$.

iv) $NVLA(P \cap Q) = NVLA(P) \cap NVLA(Q)$.

v) $NVUA(P \cup Q) = NVUA(P) \cup NVUA(Q)$.

vi) $NVUA(P \cap Q) \subseteq NVUA(P) \cap NVUA(Q)$.

vii) For $P \subseteq Q$, we have $NVLA(P) \subseteq NVLA(Q)$ and $NVUA(P) \subseteq NVUA(Q)$.

viii) $NVUA(P^c) = \left(NVLA(P)\right)^c$ and $NVLA(P^c) = \left(NVUA(P)\right)^c$.

ix) $NVUA\left(NVUA(P)\right) = NVLA\left(NVUU(P)\right) = NVUA(P)$.

x) $NVLA\left(NVLA(P)\right) = NVUA\left(NVLA(P)\right) = NVLA(P)$

Definition 3.3: Let Ω be universe, $\mathcal{R}^*$ be an equivalence relation on Ω and Q be a neutrosophic vague set in Ω and if $\tau_{\mathcal{R}^*}(Q) = \{0_{NV}, 1_{NV}, NVLA(Q), NVUA(Q), NVB(Q)\}$ where $Q \subseteq \Omega$. $\tau_{\mathcal{R}^*}(Q)$ satisfies the following axioms:

i) 0_{NV} and $1_{NV} \in \tau_{\mathcal{R}^*}(Q)$.

ii) The union of the elements of any sub-collection of $\tau_{\mathcal{R}^*}(Q)$ is in $\tau_{\mathcal{R}^*}(Q)$.

iii) The intersection of the elements of any finite sub-collection of $\tau_{\mathcal{R}^*}(Q)$ is in $\tau_{\mathcal{R}^*}(Q)$.

Then $\tau_{\mathcal{R}^*}(Q)$ is said to be neutrosophic vague nano topology. We call $(\Omega, \tau_{\mathcal{R}^*}(Q))$ as neutrosophic vague nano topological space. The elements of $\tau_{\mathcal{R}^*}(Q)$ are called as neutrosophic vague nano open sets.

Example 3.4: Let $\Omega = \{u, v, w, x, y, z\}$ be universe, $\mathcal{R}^*$ be an equivalence relation on Ω, so we have

$$\Omega = \begin{Bmatrix} \langle u: [0.6,0.9]; [0.2,0.5]; [0.1,0.4]\rangle, \langle v: [0.1,0.4]; [0.3,0.5]; [0.6,0.9]\rangle, \\ \langle w: [0.7,0.8]; [0.1,0.6]; [0.2,0.3]\rangle, \langle x: [0.2,0.6]; [0.5,0.7]; [0.4,0.8]\rangle, \\ \langle y: [0.3,0.7]; [0.6,0.8]; [0.3,0.7]\rangle, \langle z: [0.4,0.5]; [0.1,0.3]; [0.5,0.6]\rangle \end{Bmatrix}$$

${}^{\Omega}/_{\mathcal{R}^*} = \{\{u, w\}, \{v\}, \{x, y\}, \{z\}\}$ is the $\mathcal{R}^*$ be an equivalence relation on Ω. Let $Q = \{u, v, w, x, z\}$ then

$$Q = \begin{Bmatrix} \langle u: [0.6,0.9]; [0.2,0.5]; [0.1,0.4]\rangle, \langle v: [0.1,0.4]; [0.3,0.5]; [0.6,0.9]\rangle, \\ \langle w: [0.7,0.8]; [0.1,0.6]; [0.2,0.3]\rangle, \langle x: [0.2,0.6]; [0.5,0.7]; [0.4,0.8]\rangle, \\ \langle z: [0.4,0.5]; [0.1,0.3]; [0.5,0.6]\rangle \end{Bmatrix}$$

$$NVLA(Q) = \begin{Bmatrix} \langle u: [0.6,0.8]; [0.2,0.6]; [0.2,0.4]\rangle, \langle v: [0.1,0.4]; [0.3,0.5]; [0.6,0.9]\rangle, \\ \langle w: [0.6,0.8]; [0.2,0.6]; [0.2,0.4]\rangle, \langle x: [0.2,0.6]; [0.6,0.8]; [0.4,0.8]\rangle, \\ \langle z: [0.4,0.5]; [0.1,0.3]; [0.5,0.6]\rangle \end{Bmatrix}$$

$$NVUA(Q) = \begin{Bmatrix} \langle u: [0.7,0.9]; [0.1,0.5]; [0.1,0.3]\rangle, \langle v: [0.1,0.4]; [0.3,0.5]; [0.6,0.9]\rangle, \\ \langle w: [0.7,0.9]; [0.1,0.5]; [0.1,0.3]\rangle, \langle x: [0.3,0.7]; [0.5,0.7]; [0.3,0.7]\rangle, \\ \langle z: [0.4,0.5]; [0.1,0.3]; [0.5,0.6]\rangle \end{Bmatrix}$$

$$NVB(Q) = \begin{Bmatrix} \langle u: [0.2,0.4]; [0.4,0.8]; [0.6,0.8]\rangle, \langle v: [0.1,0.4]; [0.5,0.7]; [0.6,0.9]\rangle, \\ \langle w: [0.2,0.4]; [0.4,0.8]; [0.6,0.8]\rangle, \langle x: [0.3,0.7]; [0.5,0.7]; [0.3,0.7]\rangle, \\ \langle z: [0.4,0.5]; [0.7,0.9]; [0.5,0.6]\rangle \end{Bmatrix}$$

Then $\tau_{\mathcal{R}^*}(Q) = \{0_{NV}, 1_{NV}, NVLA(Q), NVUA(Q), NVB(Q)\}$ is neutrosophic vague nano topology on Ω.

Definition 3.5: Let Ω be universe and let $Q \subseteq \Omega$, then the following statements holds:

i) If $NVLA(Q) = 0_{NV}$ and $NVUA(Q) = 1_{NV}$, then $\tau_{\mathcal{R}^*}(Q) = \{0_{NV}, 1_{NV}\}$, is neutrosophic vague nano discrete topology on Ω.

ii) If $NVLA(Q) = NVUA(Q) = Q$, then the neutrosophic vague nano topology is $\tau_{\mathcal{R}^*}(Q) = \{0_{NV}, 1_{NV}, NVLA(Q)\}$.

iii) If $NVLA(Q) = 0_{NV}$ and $NVUA(Q) \neq 1_{NV}$, then $\tau_{\mathcal{R}^*}(Q) = \{0_{NV}, 1_{NV}, NVUA(Q)\}$.

iv) If $NVLA(Q) \neq 0_{NV}$ and $NVUA(Q) = 1_{NV}$, then $\tau_{\mathcal{R}^*}(Q) = \{0_{NV}, 1_{NV}, NVLA(Q), NVB(Q)\}$.

v) If $NVLA(Q) \neq NVLA(Q)$ where $NVLA(Q) \neq 0_{NV}$ and $NVUA(Q) \neq 1_{NV}$, then $\tau_{\mathcal{R}^*}(Q) = \{0_{NV}, 1_{NV}, NVLA(Q), NVUA(Q), NVB(Q)\}$ is neutrosophic vague nano discrete topology on Ω.

Definition 3.5: Let $(\Omega, \tau_{\mathcal{R}^*}(Q))$ be neutrosophic vague nano topological space with respect to Q, where $Q \subseteq \Omega$ and if $A \subseteq \Omega$, then

i) The neutrosophic vague nano interior of the set A is defined as the union of all neutrosophic vague nano open subsets contained in A, and is denoted by $NV_Q\, int(A)$.

ii) The neutrosophic vague nano closure of the set A is defined as the intersection of all neutrosophic vague nano closed subsets containing A, and is denoted by $NV_Q\, cl(A)$.

Theorem 3.6: Let $(\Omega, \tau_{\mathcal{R}^*}(Q))$ be neutrosophic vague nano topological space with respect to Q, where $Q \subseteq \Omega$. Let A and B be two neutrosophic vague nano subsets of Ω. Then the following statements hold:

i) $NV_Q\, int(A) \subseteq A$.

ii) $A \subseteq NV_Q\, cl(A)$.

iii) A is neutrosophic vague nano closed if and only if $NV_Q\, cl(A) = A$.

iv) $NV_Q\,cl(0_{NV}) = 0_{NV}$ and $NV_Q\,cl(1_{NV}) = 1_{NV}$.

v) $A \subseteq B$ implies $NV_Q\,cl(A) \subseteq NV_Q\,cl(B)$.

vi) $NV_Q\,cl(A \cup B) = NV_Q\,cl(A) \cup NV_Q\,cl(B)$.

vii) $NV_Q\,cl(A \cap B) \subseteq NV_Q\,cl(A) \cap NV_Q\,cl(B)$.

viii) $NV_Q\,cl\big(NV_Q\,cl(A)\big) = NV_Q\,cl(A)$.

Proof:

i) By definition of neutrosophic vague nano interior we have, $NV_Q\,int(A) \subseteq A$.

ii) By definition of neutrosophic vague nano closure we have, $A \subseteq NV_Q\,cl(A)$.

iii) If A is neutrosophic vague nano closed set, then A is the smallest neutrosophic vague nano closed set containing itself and hence $NV_Q\,cl(A) = A$. Conversely, if $NV_Q\,cl(A) = A$, then A is the smallest neutrosophic vague nano closed set containing itself and hence A is neutrosophic vague nano closed set.

iv) Since 0_{NV} and 1_{NV} are neutrosophic vague nano closed in $\big(\Omega, \tau_{\mathcal{R}}(Q)\big)$, so $NV_Q\,cl(0_{NV}) = 0_{NV}$ and $NV_Q\,cl(1_{NV}) = 1_{NV}$.

v) If $A \subseteq B$, since $B \subseteq NV_Q\,cl(B)$, then $A \subseteq NV_Q\,cl(B)$. That is, $NV_Q\,cl(B)$ is a neutrosophic vague nano closed set containing A. But $NV_Q\,cl(A)$ is the smallest neutrosophic vague nano closed set containing A. Therefore, $NV_Q\,cl(A) \subseteq NV_Q\,cl(B)$.

vi) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, $NV_Q\,cl(A) \subseteq NV_Q\,cl(A \cup B)$ and $NV_Q\,cl(B) \subseteq NV_Q\,cl(A \cup B)$. Therefore $NV_Q\,cl(A) \cup NV_Q\,cl(B) \subseteq NV_Q\,cl(A \cup B)$. By the fact that $A \cup B \subseteq NV_Q\,cl(A) \cup NV_Q\,cl(B)$, and since $NV_Q\,cl(A \cup B)$ is the smallest nano closed set containing $A \cup B$, so $NV_Q\,cl(A \cup B) \subseteq NV_Q\,cl(A) \cup NV_Q\,cl(B)$. Thus $NV_Q\,cl(A \cup B) = NV_Q\,cl(A) \cup NV_Q\,cl(B)$.

vii) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, $NV_Q\,cl(A \cap B) \subseteq NV_Q\,cl(A) \cap NV_Q\,cl(B)$.

viii) Since $NV_Q\,cl(A)$ is neutrosophic vague nano closed, $NV_Q\,cl\big(NV_Q\,cl(A)\big) = NV_Q\,cl(A)$.

Theorem 3.7: Let $\big(\Omega, \tau_{\mathcal{R}^*}(Q)\big)$ be neutrosophic vague nano topological space with respect to Q, where $Q \subseteq \Omega$ and if $A \subseteq \Omega$, then

i) $1_{NV} - NV_Q\,int(A) = NV_Q\,cl(1_{NV} - A)$.

ii) $1_{NV} - NV_Q\,cl(A) = NV_Q\,int(1_{NV} - A)$.

4. Real Time Application Of Neutrosophic Vague Nano Topological Space:

In this example we use the neutrosophic vague nano topology to find the vital factors of "Stroke" by using topological reduction of attributes in the data set.

We consider the following information table about patients various attributes such as Age, Sugar, Blood Pressure, Heart Disease, BMI, Family History, Smoking, Pain Killer Intake are taken as the data base set. From this data set we find the vital factor for causing stroke among patients. Here $V = \{m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, m_9, m_{10}\}$ the set of patients and S={Age, Sugar, Blood Pressure, Heart Disease, BMI, Family History, Smoking, Pain Killer Intake} the set of factors that may lead to stroke. Table 1 gives the information of the patients and in Table 2 the patients are denoted using the neutrosophic vague sets.

Patients	**Age**	**Sugar**	**Blood Pressure**	**Heart Disease**	**BMI**	**Family History**	**Smoking**	**Pain Killer Intake**	**Result**
m_1	Very Old	Yes	Very High	Yes	Over Weight	Yes	No	No	Yes
m_2	Old	Yes	Very High	Yes	Over Weight	Yes	No	No	Yes
m_3	Old	No	High	No	Obese	Yes	Yes	Yes	No
m_4	Very Old	Yes	Very High	Yes	Over Weight	Yes	No	No	Yes
m_5	Very Old	Yes	Normal	Yes	Normal	No	No	No	No
m_6	Very Old	Yes	Very High	No	Normal	No	Yes	No	Yes

m_7	Very Old	No	High	No	Obese	Yes	Yes	Yes	No
m_8	Old	No	Normal	No	Obese	No	Yes	Yes	Yes
m_9	Old	No	High	No	Obese	Yes	Yes	Yes	Yes
m_{10}	Old	No	Normal	No	Obese	Yes	Yes	Yes	No

Table 1: Patient's possible attribute

Patients	**Neutrosophic Vague Sets**
m_1	$\{\langle[0.5,0.8];[0.1,0.2];[0.2,0.5]\rangle\}$
m_2	$\{\langle[0.6,0.7];[0.3,0.5];[0.3,0.4]\rangle\}$
m_3	$\{\langle[0.2,0.3];[0.5,0.7];[0.7,0.8]\rangle\}$
m_4	$\{\langle[0.6,0.9];[0.2,0.4];[0.1,0.4]\rangle\}$
m_5	$\{\langle[0.1,0.4];[0.3,0.6];[0.6,0.9]\rangle\}$
m_6	$\{\langle[0.6,0.7];[0.4,0.5];[0.3,0.4]\rangle\}$
m_7	$\{\langle[0.1,0.2];[0.7,0.9];[0.8,0.9]\rangle\}$
m_8	$\{\langle[0.7,0.9];[0.2,0.3];[0.1,0.3]\rangle\}$
m_9	$\{\langle[0.4,0.7];[0.5,0.8];[0.3,0.6]\rangle\}$
m_{10}	$\{\langle[0.1,0.5];[0.4,0.6];[0.5,0.9]\rangle\}$

Table 2: Neutrosophic Vague sets

Here $V = \{m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, m_9, m_{10}\}$ the set of patients and S = {Age, Sugar, Blood Pressure, Heart Disease, BMI, Family History, Smoking, Pain Killer Intake} be the set of attributes that may cause stroke. In short the set is denoted by S = {AG, SU, BP, HD, BMI, FH, SM, PK}. The family of equivalence classes corresponding to S is given by $V/_{\mathcal{R}^*(S)} = \{\{m_1, m_4\}, \{m_2\}, \{m_3, m_9\}, \{m_5\}, \{m_6\}, \{m_7\}, \{m_8\}, \{m_{10}\}\}$.

Case 1: (Patients affected by Stroke)

Let $V = \{m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, m_9, m_{10}\}$ the set of patients. Let $V/_{\mathcal{R}^*(S)} = \{\{m_1, m_4\}, \{m_2\}, \{m_3, m_9\}, \{m_5\}, \{m_6\}, \{m_7\}, \{m_8\}, \{m_{10}\}\}$ be an equivalence relation on V and $P^* = \{m_1, m_2, m_4, m_6, m_8, m_9\}$ be the set of patients affected by stroke. Then

$$NVLA(P^*) = \begin{Bmatrix} \langle m_1: [.5,.8]; [.2,.4]; [.2,.5]\rangle, \langle m_2: [.6,.7]; [.3,.5]; [.3,.4]\rangle, \\ \langle m_4: [.5,.8]; [.2,.4]; [.2,.5]\rangle, \langle m_6: [.6,.7]; [.4,.5]; [.3,.4]\rangle, \\ \langle m_8: [.7,.9]; [.2,.3]; [.1,.3]\rangle, \langle m_9: [.2,.3]; [.5,.8]; [.7,.8]\rangle \end{Bmatrix}$$

$$NVUA(P^*) = \begin{Bmatrix} \langle m_1: [.6,0.9]; [.1,.2]; [.1,.4]\rangle, \langle m_2: [.6,.7]; [.3,.5]; [.3,.4]\rangle, \\ \langle m_4: [.6,0.9]; [.1,.2]; [.1,.4]\rangle, \langle m_6: [.6,.7]; [.4,.5]; [.3,.4]\rangle, \\ \langle m_8: [.7,0.9]; [.2,.3]; [.1,.3]\rangle, \langle m_9: [.4,.7]; [.5,.7]; [.3,.6]\rangle \end{Bmatrix}$$

$$NVB(P^*) = \begin{Bmatrix} \langle m_1: [.2,.5]; [.6,.8]; [.5,.8]\rangle, \langle m_2: [.3,.4]; [.5,.7]; [.6,.7]\rangle, \\ \langle m_4: [.2,.5]; [.6,.8]; [.5,.8]\rangle, \langle m_6: [.3,.4]; [.5,.6]; [.6,.7]\rangle, \\ \langle m_8: [.1,.3]; [.7,.8]; [.7,.9]\rangle, \langle m_9: [.4,.7]; [.5,.7]; [.3,.6]\rangle \end{Bmatrix}$$

Therefore, $\tau_S(P^*) = \{0_{NV}, 1_{NV}, NVLA(P^*), NVUA(P^*), NVB(P^*)\}$ ----------- (1)

Step 1: Let $P^* = \{m_1, m_2, m_4, m_6, m_8, m_9\}$, when the attribute "Age" is removed from S, we have

$V/_{\mathcal{R}^*(S-AG)} = \{\{m_1, m_2, m_4\}, \{m_3, m_7, m_9\}, \{m_5\}, \{m_6\}, \{m_8\}, \{m_{10}\}\}$. Then

$$NVLA(P^*) = \begin{Bmatrix} \langle m_1: [.5,.7]; [.3,.5]; [.3,.5]\rangle, \langle m_2: [.5,.7]; [.3,.5]; [.3,.5]\rangle, \\ \langle m_4: [.5,.7]; [.3,.5]; [.3,.5]\rangle, \langle m_6: [.6,.7]; [.4,.5]; [.3,.4]\rangle, \\ \langle m_8: [.7,.9]; [.2,.3]; [.1,.3]\rangle, \langle m_9: [.1,.2]; [.7,.9]; [.8,.9]\rangle \end{Bmatrix}$$

$$NVUA(P^*) = \begin{Bmatrix} \langle m_1: [.6,.9]; [.1,.2]; [.1,.4]\rangle, \langle m_2: [.6,.9]; [.1,.2]; [.1,.4]\rangle, \\ \langle m_4: [.6,.9]; [.1,.2]; [.1,.4]\rangle, \langle m_6: [.6,.7]; [.4,.5]; [.3,.4]\rangle, \\ \langle m_8: [.7,.9]; [.2,.3]; [.1,.3]\rangle, \langle m_9: [.4,.7]; [.5,.7]; [.3,.6]\rangle \end{Bmatrix}$$

$$NVB(P^*) = \begin{Bmatrix} \langle m_1: [.3,.5]; [.5,.7]; [.5,.7]\rangle, \langle m_2: [.3,.5]; [.5,.7]; [.5,.7]\rangle, \\ \langle m_4: [.3,.5]; [.5,.7]; [.5,.7]\rangle, \langle m_6: [.3,.4]; [.5,.6]; [.6,.7]\rangle, \\ \langle m_8: [.1,.3]; [.7,.8]; [.7,.9]\rangle, \langle m_9: [.4,.7]; [.5,.7]; [.3,.6]\rangle \end{Bmatrix}$$

Therefore, $\tau_S(P^*) = \{0_{NV}, 1_{NV}, NVLA(P^*), NVUA(P^*), NVB(P^*)\} \neq$ (1)

Step 2: Let $P^* = \{m_1, m_2, m_4, m_6, m_8, m_9\}$, when the attribute "Sugar" is removed from S, we have $V/_{\mathcal{R}^*(S-SU)} = \{\{m_1, m_4\}, \{m_2\}, \{m_3, m_9\}, \{m_5\}, \{m_6\}, \{m_7\}, \{m_8\}, \{m_{10}\}\}$. Then

$$NVLA(P^*) = \begin{Bmatrix} \langle m_1: [.5,.8]; [.2,.4]; [.2,.5]\rangle, \langle m_2: [.6,.7]; [.3,.5]; [.3,.4]\rangle, \\ \langle m_4: [.5,.8]; [.2,.4]; [.2,.5]\rangle, \langle m_6: [.6,.7]; [.4,.5]; [.3,.4]\rangle, \\ \langle m_8: [.7,.9]; [.2,.3]; [.1,.3]\rangle, \langle m_9: [.2,.3]; [.5,.8]; [.7,.8]\rangle \end{Bmatrix}$$

$$NVUA(P^*) = \begin{Bmatrix} \langle m_1 : [.6,.9]; [.1,.2]; [.1,.4] \rangle, \langle m_2 : [.6,.7]; [.3,.5]; [.3,.4] \rangle, \\ \langle m_4 : [.6,.9]; [.1,.2]; [.1,.4] \rangle, \langle m_6 : [.6,.7]; [.4,.5]; [.3,.4] \rangle, \\ \langle m_8 : [.7,.9]; [.2,.3]; [.1,.3] \rangle, \langle m_9 : [.4,.7]; [.5,.7]; [.3,.6] \rangle \end{Bmatrix}$$

$$NVB(P^*) = \begin{Bmatrix} \langle m_1 : [.2,.5]; [.6,.8]; [.5,.8] \rangle, \langle m_2 : [.3,.4]; [.5,.7]; [.6,.7] \rangle, \\ \langle m_4 : [.2,.5]; [.6,.8]; [.5,.8] \rangle, \langle m_6 : [.3,.4]; [.5,.6]; [.6,.7] \rangle, \\ \langle m_8 : [.1,.3]; [.7,.8]; [.7,.9] \rangle, \langle m_9 : [.4,.7]; [.5,.7]; [.3,.6] \rangle \end{Bmatrix}$$

Therefore, $\tau_S(P^*) = \{0_{NV}, 1_{NV}, NVLA(P^*), NVUA(P^*), NVB(P^*)\} = (1)$

Step 3: Let $P^* = \{m_1, m_2, m_4, m_6, m_8, m_9\}$, when the attribute "Blood Pressure" is removed from S, we have $V/_{\mathcal{R}^*(S - BP)} = \{\{m_1, m_4\}, \{m_2\}, \{m_3, m_9, m_{10}\}, \{m_5\}, \{m_6\}, \{m_7\}, \{m_8\}\}$. Then

$$NVLA(P^*) = \begin{Bmatrix} \langle m_1 : [.5,.8]; [.2,.4]; [.2,.5] \rangle, \langle m_2 : [.6,.7]; [.3,.5]; [.3,.4] \rangle, \\ \langle m_4 : [.5,.8]; [.2,.4]; [.2,.5] \rangle, \langle m_6 : [.6,.7]; [.4,.5]; [.3,.4] \rangle, \\ \langle m_8 : [.7,.9]; [.2,.3]; [.1,.3] \rangle, \langle m_9 : [.1,.3]; [.5,.8]; [.7,.9] \rangle \end{Bmatrix}$$

$$NVUA(P^*) = \begin{Bmatrix} \langle m_1 : [.6,.9]; [.1,.2]; [.1,.4] \rangle, \langle m_2 : [.6,.7]; [.3,.5]; [.3,.4] \rangle, \\ \langle m_4 : [.6,.9]; [.1,.2]; [.1,.4] \rangle, \langle m_6 : [.6,.7]; [.4,.5]; [.3,.4] \rangle, \\ \langle m_8 : [.7,.9]; [.2,.3]; [.1,.3] \rangle, \langle m_9 : [.4,.7]; [.4,.6]; [.3,.6] \rangle \end{Bmatrix}$$

$$NVB(P^*) = \begin{Bmatrix} \langle m_1 : [.2,.5]; [.6,.8]; [.5,.8] \rangle, \langle m_2 : [.3,.4]; [.5,.7]; [.6,.7] \rangle, \\ \langle m_4 : [.2,.5]; [.6,.8]; [.5,.8] \rangle, \langle m_6 : [.3,.4]; [.5,.6]; [.6,.7] \rangle, \\ \langle m_8 : [.1,.3]; [.7,.8]; [.7,.9] \rangle, \langle m_9 : [.4,.7]; [.4,.6]; [.3,.6] \rangle \end{Bmatrix}$$

Therefore, $\tau_S(P^*) = \{0_{NV}, 1_{NV}, NVLA(P^*), NVUA(P^*), NVB(P^*)\} \neq (1)$

Step 4: Let $P^* = \{m_1, m_2, m_4, m_6, m_8, m_9\}$, when the attribute "Heart Disease" is removed from S, we have $V/_{\mathcal{R}^*(S - HD)} = \{\{m_1, m_4\}, \{m_2\}, \{m_3, m_9\}, \{m_5\}, \{m_6\}, \{m_7\}, \{m_8\}, \{m_{10}\}\}$.

Then

$$NVLA(P^*) = \begin{Bmatrix} \langle m_1 : [.5,.8]; [.2,.4]; [.2,.5] \rangle, \langle m_2 : [.6,.7]; [.3,.5]; [.3,.4] \rangle, \\ \langle m_4 : [.5,.8]; [.2,.4]; [.2,.5] \rangle, \langle m_6 : [.6,.7]; [.4,.5]; [.3,.4] \rangle, \\ \langle m_8 : [.7,.9]; [.2,.3]; [.1,.3] \rangle, \langle m_9 : [.2,.3]; [.5,.8]; [.7,.8] \rangle \end{Bmatrix}$$

$$NVUA(P^*) = \begin{Bmatrix} \langle m_1 : [.6,.9]; [.1,.2]; [.1,.4] \rangle, \langle m_2 : [.6,.7]; [.3,.5]; [.3,.4] \rangle, \\ \langle m_4 : [.6,.9]; [.1,.2]; [.1,.4] \rangle, \langle m_6 : [.6,.7]; [.4,.5]; [.3,.4] \rangle, \\ \langle m_8 : [.7,.9]; [.2,.3]; [.1,.3] \rangle, \langle m_9 : [.4,.7]; [.5,.7]; [.3,.6] \rangle \end{Bmatrix}$$

$$NVB(P^*) = \begin{Bmatrix} \langle m_1 : [.2,.5]; [.6,.8]; [.5,.8] \rangle, \langle m_2 : [.3,.4]; [.5,.7]; [.6,.7] \rangle, \\ \langle m_4 : [.2,.5]; [.6,.8]; [.5,.8] \rangle, \langle m_6 : [.3,.4]; [.5,.6]; [.6,.7] \rangle, \\ \langle m_8 : [.1,.3]; [.7,.8]; [.7,.9] \rangle, \langle m_9 : [.4,.7]; [.5,.7]; [.3,.6] \rangle \end{Bmatrix}$$

Therefore, $\tau_S(P^*) = \{0_{NV}, 1_{NV}, NVLA(P^*), NVUA(P^*), NVB(P^*)\} = (1)$

Step 5: Let $P^* = \{m_1, m_2, m_4, m_6, m_8, m_9\}$, when the attribute "BMI" is removed from S, we have $V/_{\mathcal{R}^*(S - BMI)} = \{\{m_1, m_4\}, \{m_2\}, \{m_3, m_9\}, \{m_5\}, \{m_6\}, \{m_7\}, \{m_8\}, \{m_{10}\}\}$. Then

$$NVLA(P^*) = \begin{Bmatrix} \langle m_1: [.5,.8]; [.2,.4]; [.2,.5]\rangle, \langle m_2: [.6,.7]; [.3,.5]; [.3,.4]\rangle, \\ \langle m_4: [.5,.8]; [.2,.4]; [.2,.5]\rangle, \langle m_6: [.6,.7]; [.4,.5]; [.3,.4]\rangle, \\ \langle m_8: [.7,.9]; [.2,.3]; [.1,.3]\rangle, \langle m_9: [.2,.3]; [.5,.8]; [.7,.8]\rangle \end{Bmatrix}$$

$$NVUA(P^*) = \begin{Bmatrix} \langle m_1: [.6,.9]; [.1,.2]; [.1,.4]\rangle, \langle m_2: [.6,.7]; [.3,.5]; [.3,.4]\rangle, \\ \langle m_4: [.6,.9]; [.1,.2]; [.1,.4]\rangle, \langle m_6: [.6,.7]; [.4,.5]; [.3,.4]\rangle, \\ \langle m_8: [.7,.9]; [.2,.3]; [.1,.3]\rangle, \langle m_9: [.4,.7]; [.5,.7]; [.3,.6]\rangle \end{Bmatrix}$$

$$NVB(P^*) = \begin{Bmatrix} \langle m_1: [.2,.5]; [.6,.8]; [.5,.8]\rangle, \langle m_2: [.3,.4]; [.5,.7]; [.6,.7]\rangle, \\ \langle m_4: [.2,.5]; [.6,.8]; [.5,.8]\rangle, \langle m_6: [.3,.4]; [.5,.6]; [.6,.7]\rangle, \\ \langle m_8: [.1,.3]; [.7,.8]; [.7,.9]\rangle, \langle m_9: [.4,.7]; [.5,.7]; [.3,.6]\rangle \end{Bmatrix}$$

Therefore, $\tau_S(P^*) = \{0_{NV}, 1_{NV}, NVLA(P^*), NVUA(P^*), NVB(P^*)\} = (1)$

Step 6: Let $P^* = \{m_1, m_2, m_4, m_6, m_8, m_9\}$, when the attribute "Family History" is removed from S, we have $V/_{\mathcal{R}^*(S - FH)} = \{\{m_1, m_4\}, \{m_2\}, \{m_3, m_9\}, \{m_5\}, \{m_6\}, \{m_7\}, \{m_8, m_{10}\}\}$. Then

$$NVLA(P^*) = \begin{Bmatrix} \langle m_1: [.5,.8]; [.2,.4]; [.2,.5]\rangle, \langle m_2: [.6,.7]; [.3,.5]; [.3,.4]\rangle, \\ \langle m_4: [.5,.8]; [.2,.4]; [.2,.5]\rangle, \langle m_6: [.6,.7]; [.4,.5]; [.3,.4]\rangle, \\ \langle m_8: [.1,.5]; [.4,.6]; [.5,.9]\rangle, \langle m_9: [.2,.3]; [.5,.8]; [.7,.8]\rangle \end{Bmatrix}$$

$$NVUA(P^*) = \begin{Bmatrix} \langle m_1: [.6,.9]; [.1,.2]; [.1,.4]\rangle, \langle m_2: [.6,.7]; [.3,.5]; [.3,.4]\rangle, \\ \langle m_4: [.6,.9]; [.1,.2]; [.1,.4]\rangle, \langle m_6: [.6,.7]; [.4,.5]; [.3,.4]\rangle, \\ \langle m_8: [.7,.9]; [.2,.3]; [.1,.3]\rangle, \langle m_9: [.4,.7]; [.5,.7]; [.3,.6]\rangle \end{Bmatrix}$$

$$NVB(P^*) = \begin{Bmatrix} \langle m_1: [.2,.5]; [.6,.8]; [.5,.8]\rangle, \langle m_2: [.3,.4]; [.5,.7]; [.6,.7]\rangle, \\ \langle m_4: [.2,.5]; [.6,.8]; [.5,.8]\rangle, \langle m_6: [.3,.4]; [.5,.6]; [.6,.7]\rangle, \\ \langle m_8: [.5,.9]; [.4,.6]; [.1,.5]\rangle, \langle m_9: [.4,.7]; [.5,.7]; [.3,.6]\rangle \end{Bmatrix}$$

Therefore, $\tau_S(P^*) = \{0_{NV}, 1_{NV}, NVLA(P^*), NVUA(P^*), NVB(P^*)\} \neq (1)$

Step 7: Let $P^* = \{m_1, m_2, m_4, m_6, m_8, m_9\}$, when the attribute "Smoking" is removed from S, we have $V/_{\mathcal{R}^*(S - SM)} = \{\{m_1, m_4\}, \{m_2\}, \{m_3, m_9\}, \{m_5\}, \{m_6\}, \{m_7\}, \{m_8\}, \{m_{10}\}\}$. Then

$$NVLA(P^*) = \begin{Bmatrix} \langle m_1: [.5,.8]; [.2,.4]; [.2,.5]\rangle, \langle m_2: [.6,.7]; [.3,.5]; [.3,.4]\rangle, \\ \langle m_4: [.5,.8]; [.2,.4]; [.2,.5]\rangle, \langle m_6: [.6,.7]; [.4,.5]; [.3,.4]\rangle, \\ \langle m_8: [.7,.9]; [.2,.3]; [.1,.3]\rangle, \langle m_9: [.2,.3]; [.5,.8]; [.7,.8]\rangle \end{Bmatrix}$$

$$NVUA(P^*) = \begin{Bmatrix} \langle m_1: [.6,.9]; [.1,.2]; [.1,.4]\rangle, \langle m_2: [.6,.7]; [.3,.5]; [.3,.4]\rangle, \\ \langle m_4: [.6,.9]; [.1,.2]; [.1,.4]\rangle, \langle m_6: [.6,.7]; [.4,.5]; [.3,.4]\rangle, \\ \langle m_8: [.7,.9]; [.2,.3]; [.1,.3]\rangle, \langle m_9: [.4,.7]; [.5,.7]; [.3,.6]\rangle \end{Bmatrix}$$

$$NVB(P^*) = \left\{\begin{matrix} \langle m_1: [.2,.5]; [.6,.8]; [.5,.8]\rangle, \langle m_2: [.3,.4]; [.5,.7]; [.6,.7]\rangle, \\ \langle m_4: [.2,.5]; [.6,.8]; [.5,.8]\rangle, \langle m_6: [.3,.4]; [.5,.6]; [.6,.7]\rangle, \\ \langle m_8: [.1,.3]; [.7,.8]; [.7,.9]\rangle, \langle m_9: [.4,.7]; [.5,.7]; [.3,.6]\rangle \end{matrix}\right\}$$

Therefore, $\tau_S(P^*) = \{0_{NV}, 1_{NV}, NVLA(P^*), NVUA(P^*), NVB(P^*)\} =$ (1)

Step 8: Let $P^* = \{m_1, m_2, m_4, m_6, m_8, m_9\}$, when the attribute "Pain Killer Intake" is removed from S, we have

$V/_{\mathcal{R}^*(S-PK)} = \{\{m_1, m_4\}, \{m_2\}, \{m_3, m_9\}, \{m_5\}, \{m_6\}, \{m_7\}, \{m_8\}, \{m_{10}\}\}$. Then

$$NVLA(P^*) = \left\{\begin{matrix} \langle m_1: [.5,.8]; [.2,.4]; [.2,.5]\rangle, \langle m_2: [.6,.7]; [.3,.5]; [.3,.4]\rangle, \\ \langle m_4: [.5,.8]; [.2,.4]; [.2,.5]\rangle, \langle m_6: [.6,.7]; [.4,.5]; [.3,.4]\rangle, \\ \langle m_8: [.7,.9]; [.2,.3]; [.1,.3]\rangle, \langle m_9: [.2,.3]; [.5,.8]; [.7,.8]\rangle \end{matrix}\right\}$$

$$NVUA(P^*) = \left\{\begin{matrix} \langle m_1: [.6,.9]; [.1,.2]; [.1,.4]\rangle, \langle m_2: [.6,.7]; [.3,.5]; [.3,.4]\rangle, \\ \langle m_4: [.6,.9]; [.1,.2]; [.1,.4]\rangle, \langle m_6: [.6,.7]; [.4,.5]; [.3,.4]\rangle, \\ \langle m_8: [.7,.9]; [.2,.3]; [.1,.3]\rangle, \langle m_9: [.4,.7]; [.5,.7]; [.3,.6]\rangle \end{matrix}\right\}$$

$$NVB(P^*) = \left\{\begin{matrix} \langle m_1: [.2,.5]; [.6,.8]; [.5,.8]\rangle, \langle m_2: [.3,.4]; [.5,.7]; [.6,.7]\rangle, \\ \langle m_4: [.2,.5]; [.6,.8]; [.5,.8]\rangle, \langle m_6: [.3,.4]; [.5,.6]; [.6,.7]\rangle, \\ \langle m_8: [.1,.3]; [.7,.8]; [.7,.9]\rangle, \langle m_9: [.4,.7]; [.5,.7]; [.3,.6]\rangle \end{matrix}\right\}$$

Therefore, $\tau_S(P^*) = \{0_{NV}, 1_{NV}, NVLA(P^*), NVUA(P^*), NVB(P^*)\} =$ (1)

Thus CORE(S)={Age, Blood Pressure, Family History}.

Case 2: (Patients not affected by Stroke)

Let $V = \{m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, m_9, m_{10}\}$ the set of patients.

Let $V/_{\mathcal{R}^*(S)} = \{\{m_1, m_4\}, \{m_2\}, \{m_3, m_9\}, \{m_5\}, \{m_6\}, \{m_7\}, \{m_8\}, \{m_{10}\}\}$ be an equivalence relation on V and $Q^* = \{m_3, m_5, m_7, m_{10}\}$ be the set of patients not affected by stroke. Then

$$NVLA(Q^*) = \left\{\begin{matrix} \langle m_3: [.2,.3]; [.5,.8]; [.7,.8]\rangle, \langle m_5: [.1,.4]; [.3,.6]; [.6,.9]\rangle, \\ \langle m_7: [.1,.2]; [.7,.9]; [.8,.9]\rangle, \langle m_{10}: [.1,.5]; [.4,.6]; [.5,.9]\rangle \end{matrix}\right\}$$

$$NVUA(Q^*) = \left\{\begin{matrix} \langle m_3: [.4,.7]; [.5,.7]; [.3,.6]\rangle, \langle m_5: [.1,.4]; [.3,.6]; [.6,.9]\rangle, \\ \langle m_7: [.1,.2]; [.7,.9]; [.8,.9]\rangle, \langle m_{10}: [.1,.5]; [.4,.6]; [.5,.9]\rangle \end{matrix}\right\}$$

$$NVB(Q^*) = \left\{\begin{matrix} \langle m_3: [.4,.7]; [.5,.7]; [.3,.6]\rangle, \langle m_5: [.1,.4]; [.4,.7]; [.6,.9]\rangle, \\ \langle m_7: [.1,.2]; [.7,.9]; [.8,.9]\rangle, \langle m_{10}: [.1,.5]; [.4,.6]; [.5,.9]\rangle \end{matrix}\right\}$$

Therefore, $\tau_S(Q^*) = \{0_{NV}, 1_{NV}, NVLA(Q^*), NVUA(Q^*), NVB(Q^*)\}$ ----------- (2)

Step 1: Let $Q^* = \{m_3, m_5, m_7, m_{10}\}$, when the attribute "Age" is removed from S, we have $V/_{\mathcal{R}^*(S-AG)} = \{\{m_1, m_2, m_4\}, \{m_3, m_7, m_9\}, \{m_5\}, \{m_6\}, \{m_8\}, \{m_{10}\}\}$. Then

$$NVLA(Q^*) = \begin{Bmatrix} \langle m_3: [.1,.2]; [.7,.9]; [.8,.9]\rangle, \langle m_5: [.1,.4]; [.3,.6]; [.6,.9]\rangle, \\ \langle m_7: [.1,.2]; [.7,.9]; [.8,.9]\rangle, \langle m_{10}: [.1,.5]; [.4,.6]; [.5,.9]\rangle \end{Bmatrix}$$

$$NVUA(Q^*) = \begin{Bmatrix} \langle m_3: [.4,.7]; [.5,.7]; [.3,.6]\rangle, \langle m_5: [.1,.4]; [.3,.6]; [.6,.9]\rangle, \\ \langle m_7: [.1,.2]; [.7,.9]; [.8,.9]\rangle, \langle m_{10}: [.1,.5]; [.4,.6]; [.5,.9]\rangle \end{Bmatrix}$$

$$NVB(Q^*) = \begin{Bmatrix} \langle m_3: [.4,.7]; [.5,.7]; [.3,.6]\rangle, \langle m_5: [.1,.4]; [.4,.7]; [.6,.9]\rangle, \\ \langle m_7: [.1,.2]; [.7,.9]; [.8,.9]\rangle, \langle m_{10}: [.1,.5]; [.4,.6]; [.5,.9]\rangle \end{Bmatrix}$$

Therefore, $\tau_S(Q^*) = \{0_{NV}, 1_{NV}, NVLA(Q^*), NVUA(Q^*), NVB(Q^*)\} \neq$ (2)

Step 2: Let $Q^* = \{m_3, m_5, m_7, m_{10}\}$, when the attribute "Sugar" is removed from S, we have

$V/_{\mathcal{R}^*(S-SU)} = \{\{m_1, m_4\}, \{m_2\}, \{m_3, m_9\}, \{m_5\}, \{m_6\}, \{m_7\}, \{m_8\}, \{m_{10}\}\}$.

Then

$$NVLA(Q^*) = \begin{Bmatrix} \langle m_3: [.2,.3]; [.5,.8]; [.7,.8]\rangle, \langle m_5: [.1,.4]; [.3,.6]; [.6,.9]\rangle, \\ \langle m_7: [.1,.2]; [.7,.9]; [.8,.9]\rangle, \langle m_{10}: [.1,.5]; [.4,.6]; [.5,.9]\rangle \end{Bmatrix}$$

$$NVUA(Q^*) = \begin{Bmatrix} \langle m_3: [.4,.7]; [.5,.7]; [.3,.6]\rangle, \langle m_5: [.1,.4]; [.3,.6]; [.6,.9]\rangle, \\ \langle m_7: [.1,.2]; [.7,.9]; [.8,.9]\rangle, \langle m_{10}: [.1,.5]; [.4,.6]; [.5,.9]\rangle \end{Bmatrix}$$

$$NVB(Q^*) = \begin{Bmatrix} \langle m_3: [.4,.7]; [.5,.7]; [.3,.6]\rangle, \langle m_5: [.1,.4]; [.4,.7]; [.6,.9]\rangle, \\ \langle m_7: [.1,.2]; [.7,.9]; [.8,.9]\rangle, \langle m_{10}: [.1,.5]; [.4,.6]; [.5,.9]\rangle \end{Bmatrix}$$

Therefore, $\tau_S(Q^*) = \{0_{NV}, 1_{NV}, NVLA(Q^*), NVUA(Q^*), NVB(Q^*)\} =$ (2)

Step 3: Let $Q^* = \{m_3, m_5, m_7, m_{10}\}$, when the attribute "Blood Pressure" is removed from S, we have $V/_{\mathcal{R}^*(S-BP)} = \{\{m_1, m_4\}, \{m_2\}, \{m_3, m_9, m_{10}\}, \{m_5\}, \{m_6\}, \{m_7\}, \{m_8\}\}$. Then

$$NVLA(Q^*) = \begin{Bmatrix} \langle m_3: [.1,.3]; [.5,.8]; [.7,.9]\rangle, \langle m_5: [.1,.4]; [.3,.6]; [.6,.9]\rangle, \\ \langle m_7: [.1,.2]; [.7,.9]; [.8,.9]\rangle, \langle m_{10}: [.1,0.3]; [.5,.8]; [.7,.9]\rangle \end{Bmatrix}$$

$$NVUA(Q^*) = \begin{Bmatrix} \langle m_3: [.4,.7]; [.4,.6]; [.3,.6]\rangle, \langle m_5: [.1,.4]; [.3,.6]; [.6,.9]\rangle, \\ \langle m_7: [.1,.2]; [.7,.9]; [.8,.9]\rangle, \langle m_{10}: [.4,.7]; [.4,.6]; [.3,.6]\rangle \end{Bmatrix}$$

$$NVB(Q^*) = \begin{Bmatrix} \langle m_3: [.4,.7]; [.4,.6]; [.3,.6]\rangle, \langle m_5: [.1,.4]; [.4,.7]; [.6,.9]\rangle, \\ \langle m_7: [.1,.2]; [.7,.9]; [.8,.9]\rangle, \langle m_{10}: [.4,.7]; [.4,.6]; [.3,.6]\rangle \end{Bmatrix}$$

Therefore, $\tau_S(Q^*) = \{0_{NV}, 1_{NV}, NVLA(Q^*), NVUA(Q^*), NVB(Q^*)\} \neq$ (2)

Step 4: Let $Q^* = \{m_3, m_5, m_7, m_{10}\}$, when the attribute "Heart Disease" is removed from S, we have ${}^{V}/_{\mathcal{R}^*(S-HD)} = \{\{m_1, m_4\}, \{m_2\}, \{m_3, m_9\}, \{m_5\}, \{m_6\}, \{m_7\}, \{m_8\}, \{m_{10}\}\}$. Then

$$NVLA(Q^*) = \begin{Bmatrix} \langle m_3: [.2, .3]; [.5, .8]; [.7, .8]\rangle, \langle m_5: [.1, .4]; [.3, .6]; [.6, .9]\rangle, \\ \langle m_7: [.1, .2]; [.7, .9]; [.8, .9]\rangle, \langle m_{10}: [.1, .5]; [.4, .6]; [.5, .9]\rangle \end{Bmatrix}$$

$$NVUA(Q^*) = \begin{Bmatrix} \langle m_3: [.4, .7]; [.5, .7]; [.3, .6]\rangle, \langle m_5: [.1, .4]; [.3, .6]; [.6, .9]\rangle, \\ \langle m_7: [.1, .2]; [.7, .9]; [.8, .9]\rangle, \langle m_{10}: [.1, .5]; [.4, .6]; [.5, .9]\rangle \end{Bmatrix}$$

$$NVB(Q^*) = \begin{Bmatrix} \langle m_3: [.4, .7]; [.5, .7]; [.3, .6]\rangle, \langle m_5: [.1, .4]; [.4, .7]; [.6, .9]\rangle, \\ \langle m_7: [.1, .2]; [.7, .9]; [.8, .9]\rangle, \langle m_{10}: [.1, .5]; [.4, .6]; [.5, .9]\rangle \end{Bmatrix}$$

Therefore, $\tau_S(Q^*) = \{0_{NV}, 1_{NV}, NVLA(Q^*), NVUA(Q^*), NVB(Q^*)\} =$ (2)

Step 5: Let $Q^* = \{m_3, m_5, m_7, m_{10}\}$, when the attribute "BMI" is removed from S, we have

${}^{V}/_{\mathcal{R}^*(S-BMI)} = \{\{m_1, m_4\}, \{m_2\}, \{m_3, m_9\}, \{m_5\}, \{m_6\}, \{m_7\}, \{m_8\}, \{m_{10}\}\}$. Then

$$NVLA(Q^*) = \begin{Bmatrix} \langle m_3: [.2, .3]; [.5, .8]; [.7, .8]\rangle, \langle m_5: [.1, .4]; [.3, .6]; [.6, .9]\rangle, \\ \langle m_7: [.1, .2]; [.7, .9]; [.8, .9]\rangle, \langle m_{10}: [.1, .5]; [.4, .6]; [.5, .9]\rangle \end{Bmatrix}$$

$$NVUA(Q^*) = \begin{Bmatrix} \langle m_3: [.4, .7]; [.5, .7]; [.3, .6]\rangle, \langle m_5: [.1, .4]; [.3, .6]; [.6, .9]\rangle, \\ \langle m_7: [.1, .2]; [.7, .9]; [.8, .9]\rangle, \langle m_{10}: [.1, .5]; [.4, .6]; [.5, .9]\rangle \end{Bmatrix}$$

$$NVB(Q^*) = \begin{Bmatrix} \langle m_3: [.4, .7]; [.5, .7]; [.3, .6]\rangle, \langle m_5: [.1, .4]; [.4, .7]; [.6, .9]\rangle, \\ \langle m_7: [.1, .2]; [.7, .9]; [.8, .9]\rangle, \langle m_{10}: [.1, .5]; [.4, .6]; [.5, .9]\rangle \end{Bmatrix}$$

Therefore, $\tau_S(Q^*) = \{0_{NV}, 1_{NV}, NVLA(Q^*), NVUA(Q^*), NVB(Q^*)\} =$ (2)

Step 6: Let $Q^* = \{m_3, m_5, m_7, m_{10}\}$, when the attribute "Family History" is removed from S, we have ${}^{V}/_{\mathcal{R}^*(S-FH)} = \{\{m_1, m_4\}, \{m_2\}, \{m_3, m_9\}, \{m_5\}, \{m_6\}, \{m_7\}, \{m_8, m_{10}\}\}$. Then

$$NVLA(Q^*) = \begin{Bmatrix} \langle m_3: [.2, .3]; [.5, .8]; [.7, .8]\rangle, \langle m_5: [.1, .4]; [.3, .6]; [.6, .9]\rangle, \\ \langle m_7: [.1, .2]; [.7, .9]; [.8, .9]\rangle, \langle m_{10}: [.1, .5]; [.4, .6]; [.5, .9]\rangle \end{Bmatrix}$$

$$NVUA(Q^*) = \begin{Bmatrix} \langle m_3: [.4, .7]; [.5, .7]; [.3, .6]\rangle, \langle m_5: [.1, .4]; [.3, .6]; [.6, .9]\rangle, \\ \langle m_7: [.1, .2]; [.7, .9]; [.8, .9]\rangle, \langle m_{10}: [.7, .9]; [.2, .3]; [.1, .3]\rangle \end{Bmatrix}$$

$$NVB(Q^*) = \begin{Bmatrix} \langle m_3: [.4, .7]; [.5, .7]; [.3, .6]\rangle, \langle m_5: [.1, .4]; [.4, .7]; [.6, .9]\rangle, \\ \langle m_7: [.1, .2]; [.7, .9]; [.8, .9]\rangle, \langle m_{10}: [.5, .9]; [.4, .6]; [.1, .5]\rangle \end{Bmatrix}$$

Therefore, $\tau_S(Q^*) = \{0_{NV}, 1_{NV}, NVLA(Q^*), NVUA(Q^*), NVB(Q^*)\} \neq$ (2)

Step 7: Let $Q^* = \{m_3, m_5, m_7, m_{10}\}$, when the attribute "Smoking" is removed from S, we have

$V/_{\mathcal{R}^*(S-SM)} = \{\{m_1, m_4\}, \{m_2\}, \{m_3, m_9\}, \{m_5\}, \{m_6\}, \{m_7\}, \{m_8\}, \{m_{10}\}\}$. Then

$$NVLA(Q^*) = \begin{Bmatrix} \langle m_3: [.2, .3]; [.5, .8]; [.7, .8]\rangle, \langle m_5: [.1, .4]; [.3, .6]; [.6, .9]\rangle, \\ \langle m_7: [.1, .2]; [.7, .9]; [.8, .9]\rangle, \langle m_{10}: [.1, .5]; [.4, .6]; [.5, .9]\rangle \end{Bmatrix}$$

$$NVUA(Q^*) = \begin{Bmatrix} \langle m_3: [.4, .7]; [.5, .7]; [.3, .6]\rangle, \langle m_5: [.1, .4]; [.3, .6]; [.6, .9]\rangle, \\ \langle m_7: [.1, .2]; [.7, .9]; [.8, .9]\rangle, \langle m_{10}: [.1, .5]; [.4, .6]; [.5, .9]\rangle \end{Bmatrix}$$

$$NVB(Q^*) = \begin{Bmatrix} \langle m_3: [.4, .7]; [.5, .7]; [.3, .6]\rangle, \langle m_5: [.1, .4]; [.4, .7]; [.6, .9]\rangle, \\ \langle m_7: [.1, .2]; [.7, .9]; [.8, .9]\rangle, \langle m_{10}: [.1, .5]; [.4, .6]; [.5, .9]\rangle \end{Bmatrix}$$

Therefore, $\tau_S(Q^*) = \{0_{NV}, 1_{NV}, NVLA(Q^*), NVUA(Q^*), NVB(Q^*)\} = (2)$

Step 8: Let $Q^* = \{m_3, m_5, m_7, m_{10}\}$, when the attribute "Pain Killer Intake" is removed from S, we have $V/_{\mathcal{R}^*(S-PK)} = \{\{m_1, m_4\}, \{m_2\}, \{m_3, m_9\}, \{m_5\}, \{m_6\}, \{m_7\}, \{m_8\}, \{m_{10}\}\}$. Then

$$NVLA(Q^*) = \begin{Bmatrix} \langle m_3: [.2, .3]; [.5, .8]; [.7, .8]\rangle, \langle m_5: [.1, .4]; [.3, .6]; [.6, .9]\rangle, \\ \langle m_7: [.1, .2]; [.7, .9]; [.8, .9]\rangle, \langle m_{10}: [.1, .5]; [.4, .6]; [.5, .9]\rangle \end{Bmatrix}$$

$$NVUA(Q^*) = \begin{Bmatrix} \langle m_3: [.4, .7]; [.5, .7]; [.3, .6]\rangle, \langle m_5: [.1, .4]; [.3, .6]; [.6, .9]\rangle, \\ \langle m_7: [.1, .2]; [.7, .9]; [.8, .9]\rangle, \langle m_{10}: [.1, .5]; [.4, .6]; [.5, .9]\rangle \end{Bmatrix}$$

$$NVB(Q^*) = \begin{Bmatrix} \langle m_3: [.4, .7]; [.5, .7]; [.3, .6]\rangle, \langle m_5: [.1, .4]; [.4, .7]; [.6, .9]\rangle, \\ \langle m_7: [.1, .2]; [.7, .9]; [.8, .9]\rangle, \langle m_{10}: [.1, .5]; [.4, .6]; [.5, .9]\rangle \end{Bmatrix}$$

Therefore, $\tau_S(Q^*) = \{0_{NV}, 1_{NV}, NVLA(Q^*), NVUA(Q^*), NVB(Q^*)\} = (2)$

Thus CORE(S)={Age, Blood Pressure, Family History}.

Conclusion: By applying neutrosophic vague nano topology in the field of medical diagnosis to identify the key factor that is most important for the patient we conclude that, from the Core of both the cases, we observe that "Age", "Blood Pressure", and "Family History" are the vital factors for the Stroke. So patients with these factors must take proper frequent medical check and care to prevent themselves from stroke.

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Neutrosophic Pre-α, Semi- α & Pre- β Irresolute Functions

T.Rajesh Kannan [1], S. Chandrasekar [2*]

[1,2]PG and Research Department of Mathematics, Arignar Anna Government Arts College, Namakkal(DT), Tamil Nadu, India.

E-mail: [1]rajeshkannan03@yahoo.co.in,[2]chandrumat .chandrumat@gmail.com

* Correspondence: chandrumat@gmail.com;

Abstract: Smarandache introduced and developed interesting concepts Neutrosophic set from the Intuitionistic fuzzy sets. A.A. Salama introduced NTSs and continuity. Aim of this paper is we introduce and study the concepts Neutrosophic Pre-α, Semi- α & Pre- β Irresolute Functions and its Properties are discussed details.

Keywords: Neutrosophic Irresolute Functions, Neutrosophic Pre-α, Neutrosophic Semi- α, Neutrosophic Pre- β Irresolute Functions.

1. Introduction

Neutrosophic concepts have wide range of applications in the area of decision making Artificial Intelligence, Information Systems, Computer Science, Medicine, Applied Mathematics, Mechanics, Electrical & Electronic and, Management Science, etc,. In 1980s the international movement called paradoxism based on contradictions in science and literature, was founded by Smarandache[15,16], who then extended it to neutrosophy, based on contradictions and their neutrals. The mapping is the one of the important concept in topology. Neutrosophic sets have three kind like T Truth, F -Falsehood, I- Indeterminacy. Neutrosophic topological spaces (N-T-S) introduced by Salama [27,28]etal., by using Smarandache neutrosophy set. In this Paper new type of functions called as Neutrosophic Pre-α irresolute functions, Neutrosophic Pre-α, Semi- α and Pre-β Irresolute Functions. Also the interrelationships of these functions with the other existing functions are established. Several characterizations and some interesting properties of these classes of functions are given

2. Preliminaries

In this section, we provide basic definition and operation of Neutrosophic sets and its Results

Definition 2.1 [15,16] Let $\mathcal{X}_{\mathcal{N}}$ be a non-empty fixed set. A Neutrosophic set $\mathcal{E}_1^*$ is a object having the form

$$\mathcal{E}_1^* = \{< x, \mu_{\mathcal{E}_1^*}(x), \sigma_{\mathcal{E}_1^*}(x), \gamma_{\mathcal{E}_1^*}(x) >: x \in \mathcal{X}_{\mathcal{N}}\},$$

$\mu_{\mathcal{E}_1^*}(x)$- membership function

$\sigma_{\mathcal{E}_1^*}(x)$- indeterminacy and then

$\gamma_{\mathcal{E}_1^*}(x)$- non-membership function

Definition 2.2 [15,16]. Neutrosophic set $\mathcal{E}_1^* = \{< x, \mu_{\mathcal{E}_1^*}(x), \sigma_{\mathcal{E}_1^*}(x), \gamma_{\mathcal{E}_1^*}(x) >: x \in \mathcal{X}_\mathcal{N}\}$, on $\mathcal{X}_\mathcal{N}$ and

$\forall x \in \mathcal{X}_\mathcal{N}$

$\mathcal{E}_2^* = \{< x, \mu_{\mathcal{E}_2^*}(x), \sigma_{\mathcal{E}_2^*}(x), \gamma_{\mathcal{E}_2^*}(x) >: x \in \mathcal{X}_\mathcal{N}\}$

1. $\mathcal{E}_1^* \cap \mathcal{E}_2^* = \{< x, \mu_{\mathcal{E}_1^*}(x) \cap \mu_{\mathcal{E}_2^*}(x), \sigma_{\mathcal{E}_1^*}(x) \cap \sigma_{\mathcal{E}_2^*}(x), \gamma_{\mathcal{E}_1^*}(x) \cup \gamma_{\mathcal{E}_2^*}(x) >: x \in \mathcal{X}_\mathcal{N}\}$

2. $\mathcal{E}_1^* \cup \mathcal{E}_2^* = \{< x, \mu_{\mathcal{E}_1^*}(x) \cup \mu_{\mathcal{E}_2^*}(x), \sigma_{\mathcal{E}_1^*}(x) \cup \sigma_{\mathcal{E}_2^*}(x), \gamma_{\mathcal{E}_1^*}(x) \cap \gamma_{\mathcal{E}_2^*}(x) >: x \in \mathcal{X}_\mathcal{N}\}$

3. $\mathcal{E}_1^* \subseteq \mathcal{E}_2^* \Leftrightarrow \mu_{\mathcal{E}_1^*}(x) \leq \mu_{\mathcal{E}_2^*}(x), \sigma_{\mathcal{E}_1^*}(x) \leq \sigma_{\mathcal{E}_2^*}(x) \& \gamma_{\mathcal{E}_1^*}(x) \geq \gamma_{\mathcal{E}_2^*}(x)\}$

4. the complement of $\mathcal{E}_1^*$ is $\mathcal{E}_1^{*C} = \{< x, \gamma_{\mathcal{E}_1^*}(x), 1 - \sigma_{\mathcal{E}_1^*}(x), \mu_{\mathcal{E}_1^*}(x) >: x \in \mathcal{X}_\mathcal{N}\}$

Definition 2.3 [28]. Let $\mathcal{X}_\mathcal{N}$ be non-empty set and τ_N be the collection of Neutrosophic subsets of $\mathcal{X}_\mathcal{N}$ satisfying the following properties:

1.$0_N, 1_N \in \tau_N$

3.$T_1 \cap T_2 \in \tau_N$ for any $T_1, T_2 \in \tau_N$

4. $\cup T_i \in \tau_N$ for every $\{T_i : i \in j\} \subseteq \tau_N$

Then the space $(\mathcal{X}_\mathcal{N}, \tau_N)$ is called a Neutrosophic topological spaces (N-T-S).

The element of τ_N are called Ne.OS (Neutrosophic open set)

and its complement is Ne.CS(Neutrosophic closed set)

Example 2.4. Let $\mathcal{X}_\mathcal{N}$ ={x} and $\forall x \in \mathcal{X}_\mathcal{N}$

$A_1 = \langle x, \frac{6}{10}, \frac{6}{10}, \frac{5}{10} \rangle, A_2 = \langle x, \frac{5}{10}, \frac{7}{10}, \frac{9}{10} \rangle$

$A_3 = \langle x, \frac{6}{10}, \frac{7}{10}, \frac{5}{10} \rangle \quad , A_4 = \langle x, \frac{5}{10}, \frac{6}{10}, \frac{9}{10} \rangle$

Then the collection $\tau_N = \{0_N, A_1, A_2, A_3, A_4, 1_N\}$ is called a N-T-S on $\mathcal{X}_\mathcal{N}$.

Definition 2.5. Let $(\mathcal{X}_\mathcal{N}, \tau_N)$be a N-T-S and $\mathcal{E}_1^* = \{< x, \mu_{\mathcal{E}_1^*}(x), \sigma_{\mathcal{E}_1^*}(x), \gamma_{\mathcal{E}_1^*}(x) >: x \in \mathcal{X}_\mathcal{N}\}$ be a

Neutrosophic set in $\mathcal{X}_\mathcal{N}$. Then $\mathcal{E}_1^*$ is named as

1. Neutrosophic b closed set [20] (Ne.bCS) if Ne.cl(Ne.int($\mathcal{E}_1^*$))∩Ne.int(Ne.cl($\mathcal{E}_1^*$))⊆ $\mathcal{E}_1^*$,

3. Neutrosophic α-closed set [7] (Ne. αCS) if Ne.cl(Ne.int(Ne.cl($\mathcal{E}_1^*$)))⊆ $\mathcal{E}_1^*$,

4. Neutrosophic pre-closed set [30] (Ne.Pre-CS) if Ne.cl(Ne.int($\mathcal{E}_1^*$))⊆ $\mathcal{E}_1^*$,

5. Neutrosophic regular closed set [7] (Ne.RCS) if Ne.cl(Ne.int($\mathcal{E}_1^*$))=$\mathcal{E}_1^*$,

5. Neutrosophic semi closed set [17] (Ne.SCS) if Ne.int(Ne.cl($\mathcal{E}_1^*$))⊆ $\mathcal{E}_1^*$,

Definition 2.6.[9] $(\mathcal{X}_\mathcal{N}, \tau_N)$be a N-T-S and $\mathcal{E}_1^* = \{< x, \mu_{\mathcal{E}_1^*}(x), \sigma_{\mathcal{E}_1^*}(x), \gamma_{\mathcal{E}_1^*}(x) >: x \in \mathcal{X}_\mathcal{N}\}$ be a

Neutrosophic set in $\mathcal{X}_\mathcal{N}$.Then

Neutrosophic closure of $\mathcal{E}_1^*$ is Ne.Cl($\mathcal{E}_1^*$)=∩{H:H is a Ne.CS in $\mathcal{X}_\mathcal{N}$ and $\mathcal{E}_1^*$ ⊆H}

Neutrosophic interior of $\mathcal{E}_1^*$ is Ne.Int($\mathcal{E}_1^*$)=∪{M:M is a Ne.OS in $\mathcal{X}_\mathcal{N}$ and M⊆ $\mathcal{E}_1^*$}.

Definition 2.7. Let $(\mathcal{X}_\mathcal{N}, \mathcal{T}_\mathcal{N})$ be an NTS and be an NS in $\mathcal{X}_\mathcal{N}$.

The Neutrosophic β-closure &β-interior of A are defined by

(i) $\mathcal{N}\beta cl(\mathcal{E}_1^*) = \cap\{\mathcal{E}_3^* : \mathcal{E}_3^*$ is a βCS in $\mathcal{X}_\mathcal{N}$ and $\mathcal{E}_3^* \supseteq \mathcal{E}_1^*\}$;

(ii) $\mathcal{N}\beta\text{int}(\mathcal{E}_1^*) = \cup\{\mathcal{E}_4^*: \mathcal{E}_4^*$ is a $\mathcal{N}\beta OS$ in $\mathcal{X}_{\mathcal{N}}$ and $\mathcal{E}_4^* \subseteq \mathcal{E}_1^*\}$.

Lemma 2.8.

Let $\mathcal{E}_1^*$ be an NS in NTS $(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}})$.Then

(i) $\text{Nint}(\mathcal{E}_1^*) \subseteq \text{NP int}(\mathcal{E}_1^*) \subseteq \mathcal{E}_1^* \subseteq \text{NPcl}(\mathcal{E}_1^*) \subseteq \text{Ncl}(\mathcal{E}_1^*)$

(ii) $\text{Nint}(\mathcal{E}_1^*) \subseteq \text{N}\alpha\text{int}(\mathcal{E}_1^*) \subseteq \mathcal{E}_1^* \subseteq \text{N}\alpha\text{cl}(\mathcal{E}_1^*) \subseteq \text{Ncl}(\mathcal{E}_1^*)$

(iii) $\text{Nint}(\mathcal{E}_1^*)\subseteq\text{NSint}(\mathcal{E}_1^*)\subseteq\mathcal{E}_1^*\subseteq\text{NScl}(\mathcal{E}_1^*)\subseteq\text{Ncl}(\mathcal{E}_1^*)$

(iv) $\text{Nint}(\mathcal{E}_1^*)\subseteq\text{N}\beta\text{int}(\mathcal{E}_1^*)\subseteq \mathcal{E}_1^* \subseteq \text{N}\beta\text{cl}(\mathcal{E}_1^*)\subseteq\text{Ncl}(\mathcal{E}_1^*)$.

Proof: It is easy to prove.

3.Neutrosophic Pre-α, Semi- α & Pre- β Irresolute Functions

In this section Neutrosophic pre-α-irresolute, semi-α-irresolute, Neutrosophic pre-β-irresolute functions are defined. Also, the relationships of these functions with the other existing functions are studied.

Definition 3.1.

A function $\ddot{f}: (\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}}) \longrightarrow (\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$ from an NTS $(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}})$ to another NTS $(\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$ is named as Neutrosophic β-irresolute if $\ddot{f}^{-1}(\mathcal{E}_2^*)$is a $\mathcal{N}\beta OS$ in $(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}})$ for each $\mathcal{N}\beta OS$ $\mathcal{E}_2^*$ in $(\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$.

Definition 3.2 A function$\ddot{f}: (\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}}) \longrightarrow (\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$ from an NTS $(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}})$ to another NTS $(\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$ is named as Neutrosophic pre-α-irresolute if $\ddot{f}^{-1}(\mathcal{E}_2^*)$ is an NPOS in $(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}})$ for each NαOS $\mathcal{E}_2^*$ in $(\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$.

Definition 3.3 A function $\ddot{f}: (\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}}) \longrightarrow (\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$ from an NTS $(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}})$ to another NTS $(\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$ is named as Neutrosophic α-irresolute if $\ddot{f}^{-1}(\mathcal{E}_2^*))$ is a NαOS in $(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}})$ for each NαOS $\mathcal{E}_2^*$ in $(\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$.

Definition 3.4 A function $\ddot{f}: (\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}}) \longrightarrow (\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$ from an NTS $(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}})$ to another NTS $(\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$ is named as Neutrosophic semi-α-irresolute if $\ddot{f}^{-1}(\mathcal{E}_2^*))$ is an NSOS in $(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}})$ for each NαOS $\mathcal{E}_2^*$ in $(\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$.

Definition 3.5 A function $\ddot{f}: (\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}}) \longrightarrow (\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$ from an NTS $(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}})$ to another NTS $(\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$ is named as Neutrosophic pre-β-irresolute if $\ddot{f}^{-1}(\mathcal{E}_2^*))$ is a NPOS in $(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}})$ for each $\mathcal{N}\beta OS$ $\mathcal{E}_2^*$ i$\mathcal{X}_{\mathcal{N}}$n $(\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$.

Proposition 3.6 Every Nα-irresolute function is Npre-α (NSemi-α,resp.)-irresolute function.

Proof: Let $\ddot{f}: (\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}}) \longrightarrow (\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$ be Nα-irresolute function from NTS$(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}})$ to another NTS $(\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$. Let $\mathcal{E}_2^*$ be NαOS in $\mathcal{Y}_{\mathcal{N}}$. Since $\ddot{f}$ is Nα-irresolute function,$\ddot{f}^{-1}(\mathcal{E}_2^*))$ is NαOS in $\mathcal{X}_{\mathcal{N}}$. Every N$\alpha$OS is NPOS (NSOS, resp.). So $\ddot{f}^{-1}(\mathcal{E}_2^*))$ is NPOS (NSOS, resp.) in $\mathcal{X}_{\mathcal{N}}$. Hence $\ddot{f}$ is Npre-α (NSemi-α, resp.)-irresolute function.

Proposition 3.7 Every Npre-β-irresolute function is Npre-α-(Npre, resp.) irresolute function.

Proof: Let $\ddot{f}: (\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}}) \longrightarrow (\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$ be Npre-β irresolute function from NTS $(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}})$ to another NTS $(\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$. Let $\mathcal{E}_2^*$ be NαOS (NPOS resp.) in $\mathcal{Y}_{\mathcal{N}}$. Every N$\alpha$OS (NPOS, resp.) is $\mathcal{N}\beta OS$. Since $\ddot{f}$ is Npre-β-irresolute function,$\ddot{f}^{-1}(\mathcal{E}_2^*))$ is NPOS in $\mathcal{X}_{\mathcal{N}}$. Hence $\ddot{f}$ is Npre-α-(Npre, resp.) irresolute function.

Proposition 3.8 Every Npre-β-irresolute function is $\mathcal{N}\beta$-irresolute function.

Proof: Let $\ddot{f}: (\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}}) \longrightarrow (\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$ be Npre-β irresolute function from NTS $(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}})$ to another NTS$(\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$. Let $\mathcal{E}_2^*$ be βOS . Since $\ddot{f}$ is Npre-β- irresolute function, $\ddot{f}^{-1}(\mathcal{E}_2^*))$ is NPOS in $\mathcal{X}_{\mathcal{N}}$. As every NPOS is $\mathcal{N}\beta OS$, $\ddot{f}^{-1}(\mathcal{E}_2^*))$ is $\mathcal{N}\beta OS$ in $\mathcal{X}_{\mathcal{N}}$. Hence $\ddot{f}$ is $\mathcal{N}\beta$ -irresolute function.

Proposition 3.9 Every Nirresolute function is NS-α-irresolute function.

Proof: Let $\ddot{f}$: $(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}}) \longrightarrow (\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$ be Nirresolute function from NTS$(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}})$ to another NTS $(\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$. Let $\mathcal{E}_2^*$ be NαOS in $\mathcal{Y}_{\mathcal{N}}$. Every NαOS is NSOS. Since $\ddot{f}$ is Nirresolute function, $\ddot{f}^{-1}(\mathcal{E}_2^*)$) is NSOS in $\mathcal{X}_{\mathcal{N}}$. Hence $\ddot{f}$ is NS-α- irresolute function.

Example 3.10 Let $\mathcal{X}_{\mathcal{N}}$={a,b} $\mathcal{Y}_{\mathcal{N}}$={c,d} and $\mathcal{T}_{\mathcal{N}}$= {0, $\mathcal{E}_1^*$, 1}, $\Gamma_{\mathcal{N}}$= {0, $\mathcal{E}_2^*$, 1},are NTS on $\mathcal{X}_{\mathcal{N}}$ and $\mathcal{Y}_{\mathcal{N}}$ respectively where

$$\mathcal{E}_1^* = \langle x, \left(\frac{6}{10},\frac{5}{10},\frac{4}{10}\right), \left(\frac{5}{10},\frac{5}{10},\frac{5}{10}\right)\rangle.$$

$$\mathcal{E}_2^* = \langle y, \left(\frac{2}{10},\frac{5}{10},\frac{6}{10}\right), \left(\frac{4}{10},\frac{5}{10},\frac{5}{10}\right)\rangle$$

Define an Neutrosophic function$\ddot{f}$: $(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}}) \longrightarrow (\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$. By $\ddot{f}$ (a)=d, $\ddot{f}$ (b)=c $\mathcal{E}_2^*$ is a NOS in $(\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$. So $\mathcal{E}_2^*$ is NαOS, NPOS, and $\mathcal{N}\beta OS$ in $\mathcal{Y}_{\mathcal{N}}$.

, since$\ddot{f}^{-1}(\mathcal{E}_2^*) = \langle x, \left(\frac{4}{10},\frac{5}{10},\frac{5}{10}\right), \left(\frac{2}{10},\frac{5}{10},\frac{6}{10}\right)\rangle$ is an NPOS in $\mathcal{X}_{\mathcal{N}}$

$$\ddot{f}^{-1}(\mathcal{E}_2^*) \subseteq Nint\left(Ncl\left(\ddot{f}^{-1}(\mathcal{E}_2^*)\right)\right) = 1_N$$

Also $\ddot{f}^{-1}(\mathcal{E}_2^*) \subseteq Ncl\left(Nint\left(Ncl\left(\ddot{f}^{-1}(\mathcal{E}_2^*)\right)\right)\right) = 1_N$

So $\ddot{f}^{-1}(\mathcal{E}_2^*)$ is a $\mathcal{N}\beta OS$ in $\mathcal{X}_{\mathcal{N}}$. Thus $\ddot{f}$ is Npre-β-irresolute, Npre irresolute function, Npre-α-irresolute function and $\mathcal{N}\beta$ -irresolute function.Also $\ddot{f}$ is a N precontinuous and $\mathcal{N}\beta$ -continuous. As $Nint\left(Ncl\left(Nint\left(\ddot{f}^{-1}(\mathcal{E}_2^*)\right)\right)\right) = 0_N$,$\ddot{f}^{-1}(\mathcal{E}_2^*) \nsubseteq Ncl\left(Nint\left(Ncl\left(\ddot{f}^{-1}(\mathcal{E}_2^*)\right)\right)\right)$

$\ddot{f}^{-1}(\mathcal{E}_2^*)$ is not NαOS in $\mathcal{X}_{\mathcal{N}}$. Also $\ddot{f}^{-1}(\mathcal{E}_2^*) \nsubseteq Ncl\left(Nint\left(\ddot{f}^{-1}(\mathcal{E}_2^*)\right)\right.$=$0_N$.implies $\ddot{f}^{-1}(\mathcal{E}_2^*)$ is not NSOS in $\mathcal{X}_{\mathcal{N}}$. Thus $\ddot{f}$ is not Nα-irresolute function, not NSemi-α-irresolute function, not Nα-continuous, not NSemi continuous, and not Nirresolute function.

Example 3.11

Let $\mathcal{X}_{\mathcal{N}}$={a,b} $\mathcal{Y}_{\mathcal{N}}$={c,d} and $\mathcal{T}_{\mathcal{N}}$= {0, $\mathcal{E}_1^*$, 1}, $\Gamma_{\mathcal{N}}$= {0, $\mathcal{E}_2^*$, 1},are NTS on $\mathcal{X}_{\mathcal{N}}$ and $\mathcal{Y}_{\mathcal{N}}$ respectively is a NS in $\mathcal{Y}_{\mathcal{N}}$.

$$\mathcal{E}_1^* = \langle x, \left(\frac{2}{10},\frac{5}{10},\frac{3}{10}\right), \left(\frac{4}{10},\frac{5}{10},\frac{4}{10}\right)\rangle.$$

$$\mathcal{E}_2^* = \langle y, \left(\frac{4}{10},\frac{5}{10},\frac{4}{10}\right), \left(\frac{2}{10},\frac{5}{10},\frac{3}{10}\right)\rangle$$

$$\mathcal{E}_3^* = \langle y, \left(\frac{4}{10},\frac{5}{10},\frac{4}{10}\right), \left(\frac{3}{10},\frac{5}{10},\frac{2}{10}\right)\rangle$$

is a NS in $\mathcal{Y}_{\mathcal{N}}$.

Define a Neutrosophic function $\ddot{f}$: $(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}}) \longrightarrow (\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$ by $\ddot{f}$ (a)=d, $\ddot{f}$ (b)=c $\mathcal{E}_2^*$ is a NOS in $(\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$. Also $\mathcal{E}_2^*$ is NαOS, NPOS and NSOS in $\mathcal{Y}_{\mathcal{N}}$.

$$\ddot{f}^{-1}(\mathcal{E}_2^*) = \langle x, \left(\frac{2}{10},\frac{5}{10},\frac{3}{10}\right), \left(\frac{4}{10},\frac{5}{10},\frac{4}{10}\right)\rangle$$

and $\ddot{f}^{-1}(\mathcal{E}_2^*) \subseteq Nint\left(Ncl\left(\ddot{f}^{-1}(\mathcal{E}_2^*)\right)\right) = \mathcal{E}_1^*$= . So $\ddot{f}^{-1}(\mathcal{E}_2^*) \subseteq Nint\left(Ncl\left(Nint\left(\ddot{f}^{-1}(\mathcal{E}_2^*)\right)\right)\right.$ This implies $\ddot{f}^{-1}(\mathcal{E}_2^*)$is a NαOS in $\mathcal{X}_{\mathcal{N}}$. Also $\ddot{f}^{-1}(\mathcal{E}_2^*)$ is NPOS and NSOS in $\mathcal{X}_{\mathcal{N}}$. Hence $\ddot{f}$ is a Nα-irresolute function, NS-α-irresolute function, Npre-α-irresolute function, Nα-continuous, NSemicontinuous, and Nprecontinuous.$\mathcal{E}_3^* \subseteq$ Ncl(Nint($\mathcal{E}_3^*$) $= \overline{\mathcal{E}_2^*}$. So $\mathcal{E}_3^*$ is a NSOS in $\mathcal{Y}_{\mathcal{N}}$.

Also $\ddot{f}^{-1}(\mathcal{E}_3^*) = \langle x, \left(\frac{3}{10}, \frac{5}{10}, \frac{2}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{4}{10}\right)\rangle$.Then$\ddot{f}^{-1}(\mathcal{E}_3^*) \nsubseteq Nint(Ncl(Nint\left(\ddot{f}^{-1}(\mathcal{E}_3^*)\right)) = \mathcal{E}_1^*$

.Hence$\ddot{f}^{-1}(\mathcal{E}_3^*)$ is not NαOS in $\mathcal{X}_{\mathcal{N}}$. Thus $\ddot{f}$ is not Nstrongly α-continuous.

Example 3.12 Let $\mathcal{X}_{\mathcal{N}}$={a,b} $\mathcal{Y}_{\mathcal{N}}$={c,d} and $\mathcal{T}_{\mathcal{N}}$= {0, $\mathcal{E}_1^*$, 1}, $\Gamma_{\mathcal{N}}$= {0, $\mathcal{E}_2^*$, 1},are NTS on $\mathcal{X}_{\mathcal{N}}$ and $\mathcal{Y}_{\mathcal{N}}$ respectively, where

$$\mathcal{E}_1^* = \langle x, \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle.$$

$$\mathcal{E}_2^* = \langle y, \left(\frac{2}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle$$

Define a Neutrosophic function$\ddot{f}$: ($\mathcal{X}_{\mathcal{N}}$,$\mathcal{T}_{\mathcal{N}}$) $\longrightarrow$ ($\mathcal{Y}_{\mathcal{N}}$,$\mathcal{G}_{\mathcal{N}}$). By $\ddot{f}$ (a)=d, $\ddot{f}$ (b)=c. $\mathcal{E}_2^*$ is a NOS in $\mathcal{G}_{\mathcal{N}}$ .Hence $\mathcal{E}_2^*$ is NαOS, NPOS, NSOS and $\mathcal{N}\beta OS$ in ($\mathcal{Y}_{\mathcal{N}}$,$\mathcal{G}_{\mathcal{N}}$).

$$\ddot{f}^{-1}(\mathcal{E}_2^*) = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle$$

$\ddot{f}^{-1}(\mathcal{E}_2^*)\subseteq$ Ncl(Nint($\ddot{f}^{-1}(\mathcal{E}_2^*)$)) $= \overline{\mathcal{E}_1^*}$ implies$\ddot{f}^{-1}(\mathcal{E}_2^*)$ is a NSOS in $\mathcal{X}_{\mathcal{N}}$. Also $\ddot{f}^{-1}(\mathcal{E}_2^*)$is a $\mathcal{N}\beta OS$ in $\mathcal{X}_{\mathcal{N}}$, since $\ddot{f}^{-1}(\mathcal{E}_2^*) \subseteq$ Ncl(Nint$\left(\ddot{f}^{-1}(\mathcal{E}_2^*)\right) = \overline{\mathcal{E}_1^*}$.Hence $\ddot{f}$ is Nirresolute function, NS-α-irresolute function, NSemi continuous and $\mathcal{N}\beta$ -continuous. Nint(Ncl (Nint($\ddot{f}^{-1}(\mathcal{E}_2^*)$) $= \mathcal{E}_1^*$. So $\ddot{f}^{-1}(\mathcal{E}_2^*) \nsubseteq$ Nint(Ncl (Nint($\ddot{f}^{-1}(\mathcal{E}_2^*)$)). Hence $\ddot{f}^{-1}(\mathcal{E}_2^*)$ is not NαOS in $\mathcal{X}_{\mathcal{N}}$. Also $\ddot{f}^{-1}(\mathcal{E}_2^*)$ $\nsubseteq$Nint(Ncl($\ddot{f}^{-1}(\mathcal{E}_2^*)$)). Hence $\ddot{f}^{-1}(\mathcal{E}_2^*)$ is not NPOS in $\mathcal{X}_{\mathcal{N}}$. Thus $\ddot{f}$ is not Nα- irresolute function, not Npre-α-irresolute function, not Npre irresolute function, not Npre-β-irresolute function, not Nα-continuous and not Npre continuous.

Example 3.13

Let $\mathcal{X}_{\mathcal{N}}$ = {a,b,c} = $\mathcal{Y}_{\mathcal{N}}$ and $\mathcal{T}_{\mathcal{N}}$ = {0_N, 1_N , $\mathcal{E}_1^*$, $\mathcal{E}_1^* \cup \mathcal{E}_2^*$, $\mathcal{E}_1^* \cap \mathcal{E}_2^*$},$\Gamma_{\mathcal{N}}$ = {0_N, 1_N, $\mathcal{E}_3^*$} are NTS on $\mathcal{X}_{\mathcal{N}}$ and $\mathcal{Y}_{\mathcal{N}}$ where

$$\mathcal{E}_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right)\rangle.$$

$$\mathcal{E}_2^* = \langle x, \left(\frac{2}{10}, \frac{5}{10}, \frac{7}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$$

$$\mathcal{E}_3^* = \langle y, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right)\rangle$$

$$\mathcal{E}_4^* = \langle y, \left(\frac{5}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right)\rangle$$

is a NS in $\mathcal{Y}_{\mathcal{N}}$. Define an identity Neutrosophic function $\ddot{f}$: ($\mathcal{X}_{\mathcal{N}}$,$\mathcal{T}_{\mathcal{N}}$) $\longrightarrow$ ($\mathcal{Y}_{\mathcal{N}}$,$\mathcal{G}_{\mathcal{N}}$). $\mathcal{E}_3^*$ is a NOS in $\mathcal{Y}_{\mathcal{N}}$.and $\ddot{f}^{-1}(\mathcal{E}_3^*) = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right)\rangle$

Nint(Ncl(Nint($\ddot{f}^{-1}(\mathcal{E}_3^*)$) $= \mathcal{E}_1^* \cup \mathcal{E}_2^*$.Thus $\ddot{f}^{-1}(\mathcal{E}_3^*)$. $\subseteq$ Nint(Ncl(Nint($\ddot{f}^{-1}(\mathcal{E}_3^*)$)Hence $\ddot{f}^{-1}(\mathcal{E}_3^*)$is a N$\alpha$OS in ($\mathcal{X}_{\mathcal{N}}$,$\mathcal{T}_{\mathcal{N}}$). Also $\ddot{f}^{-1}(\mathcal{E}_3^*)$ is NPOS, NSOS and $\mathcal{N}\beta OS$ in $\mathcal{X}_{\mathcal{N}}$. Therefore $\ddot{f}$ is Nα-continuous, Npre continuous, NSemicontinuous and $\mathcal{N}\beta$ -continuous. $\mathcal{E}_4^*$ is a NS in $\mathcal{Y}_{\mathcal{N}}$ and

$\mathcal{E}_4^* \subseteq$ Nint(Ncl(Nint($\mathcal{E}_4^*$) $= 1_N$. Hence $\mathcal{E}_4^*$ is a NαOS in $\mathcal{Y}_{\mathcal{N}}$. Also $\mathcal{E}_4^*$ is NPOS, NSOS and $\mathcal{N}\beta OS$ in $\mathcal{Y}_{\mathcal{N}}$.

$$\ddot{f}^{-1}(\mathcal{E}_4^*) = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right)\rangle$$

And $\ddot{f}^{-1}(\mathcal{E}_4^*) \subseteq \mathrm{Ncl}\left(\mathrm{Nint}\left(\ddot{f}^{-1}(\mathcal{E}_4^*)\right)\right) = \overline{\mathcal{E}_2^*}$.Hence$\ddot{f}^{-1}(\mathcal{E}_4^*)$. Hence $\ddot{f}^{-1}(\mathcal{E}_4^*)$is NSOS and also $\mathcal{N}\beta OS$ in $\mathcal{X}_{\mathcal{N}}$. So $\ddot{f}$ is Nirresolute function, NS-α-irresolute function and $\mathcal{N}\beta$ –irresolute function. Since $\ddot{f}^{-1}(\mathcal{E}_4^*) \not\subseteq$Nint(Ncl Nint($\ddot{f}^{-1}(\mathcal{E}_4^*)$)) = $\mathcal{E}_1^* \cup \mathcal{E}_2^*$, $\ddot{f}^{-1}(\mathcal{E}_4^*)$ is not NαOS in $\mathcal{X}_{\mathcal{N}}$ and $\ddot{f}^{-1}(\mathcal{E}_4^*) \not\subseteq$ Nint(Ncl($\ddot{f}^{-1}(\mathcal{E}_4^*)$ = $\mathcal{E}_1^* \cup \mathcal{E}_2^*$, $\ddot{f}^{-1}(\mathcal{E}_4^*)$ is not NPOS in $\mathcal{X}_{\mathcal{N}}$. Thus $\ddot{f}$ is not Nα-irresolute function, not Npre-α-irresolute function and not Npre-β-irresolute function.

Example 3.14

Let $\mathcal{X}_{\mathcal{N}}$={a,b} $\mathcal{Y}_{\mathcal{N}}$={c,d} and $\mathcal{T}_{\mathcal{N}}$= {0, $\mathcal{E}_1^*$, 1}, $\Gamma_{\mathcal{N}}$= {0, $\mathcal{E}_2^*$, 1},are NTS on $\mathcal{X}_{\mathcal{N}}$ and $\mathcal{Y}_{\mathcal{N}}$ respectively where

$$\mathcal{E}_1^* = \langle x, \left(\frac{3}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle$$

$$\mathcal{E}_2^* = \langle y, \left(\frac{4}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{3}{10}\right)\rangle$$

$$\text{And } \mathcal{E}_3^* = \langle y, \left(\frac{4}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{2}{10}\right)\rangle$$

is a NS in $\mathcal{Y}_{\mathcal{N}}$ Define a Neutrosophic function $\ddot{f}$: ($\mathcal{X}_{\mathcal{N}}$,$\mathcal{T}_{\mathcal{N}}$) ⟶ ($\mathcal{Y}_{\mathcal{N}}$,$\mathcal{G}_{\mathcal{N}}$). By $\ddot{f}$ (a)=d, $\ddot{f}$ (b)=c $\mathcal{E}_2^*$ is a NOS in ($\mathcal{Y}_{\mathcal{N}}$,$\mathcal{G}_{\mathcal{N}}$). And $\ddot{f}^{-1}(\mathcal{E}_2^*) = \langle y, \left(\frac{5}{10}, \frac{5}{10}, \frac{3}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle$ and

Ncl(Nint(($\ddot{f}^{-1}(\mathcal{E}_2^*)$))) $= \overline{\mathcal{E}_1^*}$. Thus $\ddot{f}^{-1}(\mathcal{E}_2^*) \subseteq$ Ncl(Nint(($\ddot{f}^{-1}(\mathcal{E}_2^*)$))). Hence $\ddot{f}^{-1}(\mathcal{E}_2^*)$ is an NSOS in $\mathcal{X}_{\mathcal{N}}$, which implies $\ddot{f}$ is NSemi continuous and also $\ddot{f}$ is $\mathcal{N}\beta$ -continuous. $\mathcal{E}_3^*$ is a NS in $\mathcal{Y}_{\mathcal{N}}$. Also $\mathcal{E}_3^* \subseteq$Nint(Ncl(Nint($\mathcal{E}_3^*$)= $1_{\mathcal{N}}$which implies $\mathcal{E}_3^*$ is a NαOS in $\mathcal{Y}_{\mathcal{N}}$. Hence $\mathcal{E}_3^*$ is NPOS, NSOS and $\mathcal{N}\beta OS$ in $\mathcal{Y}_{\mathcal{N}}$.$\ddot{f}^{-1}(\mathcal{E}_3^*) = \langle y, \left(\frac{5}{10}, \frac{5}{10}, \frac{3}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle$ So $\ddot{f}^{-1}(\mathcal{E}_3^*)$is a NPOS and $\mathcal{N}\beta OS$ in $\mathcal{X}_{\mathcal{N}}$.Thus $\ddot{f}$ is Npre-α-irresolute function, Npre-β-irresolute function and $\mathcal{N}\beta$ -irresolute function. Since $\ddot{f}^{-1}(\mathcal{E}_3^*) \not\subseteq$ Nint(Ncl (Nint($\ddot{f}^{-1}(\mathcal{E}_3^*)$= $\mathcal{E}_1^*$, $\ddot{f}^{-1}(\mathcal{E}_3^*)$ is not NαOS in $\mathcal{X}_{\mathcal{N}}$. Also $\ddot{f}^{-1}(\mathcal{E}_3^*) \not\subseteq$ Ncl(Nint($\ddot{f}^{-1}(\mathcal{E}_3^*)$ =$\overline{\mathcal{E}_1^*}$. So $\ddot{f}^{-1}(\mathcal{E}_3^*)$is not NSOS in $\mathcal{X}_{\mathcal{N}}$. Hence $\ddot{f}$ is not Nα-irresolute function, not Nirresolute function,and not NS-α-irresolute function.

Example 3.15

Let $\mathcal{X}_{\mathcal{N}}$ = {a,b} = $\mathcal{Y}_{\mathcal{N}}$ and

$\mathcal{T}_{\mathcal{N}}$ = {0_N, 1_N , $\mathcal{E}_1^*$, $\mathcal{E}_1^* \cup \mathcal{E}_2^*$, $\mathcal{E}_1^* \cap \mathcal{E}_2^*$}

,$\Gamma_{\mathcal{N}}$ = {0_N, 1_N, $\mathcal{E}_3^*$} are NTS on $\mathcal{X}_{\mathcal{N}}$ and $\mathcal{Y}_{\mathcal{N}}$ where

$$\mathcal{E}_1^* = \langle x, \left(\frac{2}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{2}{10}, \frac{5}{10}, \frac{4}{10}\right)\rangle.$$

$$\mathcal{E}_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right)\rangle$$

$$\mathcal{E}_3^* = \langle y, \left(\frac{2}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right)\rangle$$

$$\mathcal{E}_4^* = \langle y, \left(\frac{3}{10}, \frac{5}{10}, \frac{3}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{5}{10}\right), \rangle$$

is a NS in $\mathcal{Y}_{\mathcal{N}}$. Define an identity Neutrosophic function $\ddot{f}$: $(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}}) \longrightarrow (\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$. $\mathcal{E}_3^*$ is a NOS in $\mathcal{Y}_{\mathcal{N}}$.. $\mathcal{E}_3^*$ is a NOS, NαOS, NPOS in $(\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$. $\ddot{f}^{-1}(\mathcal{E}_3^*) = \langle y, \left(\frac{2}{10},\frac{5}{10},\frac{6}{10}\right),\left(\frac{4}{10},\frac{5}{10},\frac{6}{10}\right)\rangle$

So $\ddot{f}^{-1}(\mathcal{E}_3^*)\subseteq$Nint(Ncl(Nint($\ddot{f}^{-1}(\mathcal{E}_3^*)$) $= \mathcal{E}_1^*\cup\mathcal{E}_2^*$. Thus $\ddot{f}^{-1}(\mathcal{E}_3^*)$is a N$\alpha$OS in $\mathcal{X}_{\mathcal{N}}$.. Hence $\ddot{f}^{-1}(\mathcal{E}_3^*)$is NPOS and NSOS in $\mathcal{X}_{\mathcal{N}}$. Thus $\ddot{f}$ is Nα-irresolute, Nsemi-α-irresolute and Npre-α-irresolute function, Nα-continuous, Nprecontinuous and NSemi continuous. $\mathcal{E}_4^*$ is a NS in $\mathcal{Y}_{\mathcal{N}}$ and Ncl(Nint($\ddot{f}^{-1}(\mathcal{E}_4^*)$)) $= \overline{\mathcal{E}_3^*}$.Hence $\mathcal{E}_4^*$ $\subseteq$Ncl (Nint($\mathcal{E}_4^*$). Thus $\mathcal{E}_4^*$ is a NSOS in $\mathcal{Y}_{\mathcal{N}}$.

$$\ddot{f}^{-1}(\mathcal{E}_4^*) = \langle x, \left(\frac{3}{10},\frac{5}{10},\frac{3}{10}\right),\left(\frac{4}{10},\frac{5}{10},\frac{5}{10}\right)\rangle$$

and Ncl(Nint($\ddot{f}^{-1}(\mathcal{E}_4^*)$)) $= \overline{\mathcal{E}_1^* \cup \mathcal{E}_2^*}$

. So $\ddot{f}^{-1}(\mathcal{E}_4^*) \nsubseteq$ Ncl(Nint($\ddot{f}^{-1}(\mathcal{E}_4^*)$)). Thus $\ddot{f}^{-1}(\mathcal{E}_4^*)$is not NSOS in $\mathcal{X}_{\mathcal{N}}$.Hence $\ddot{f}$ is not Nirresolute function.

Example 3.16

Let $\mathcal{X}_{\mathcal{N}}$ = {a,b,c} = $\mathcal{Y}_{\mathcal{N}}$ and $\mathcal{T}_{\mathcal{N}}$ = {0_N, 1_N, $\mathcal{E}_1^*$},$\Gamma_{\mathcal{N}}$ = {0_N, 1_N, $\mathcal{E}_2^*$} are NTS on $\mathcal{X}_{\mathcal{N}}$ and $\mathcal{Y}_{\mathcal{N}}$ where

$$\mathcal{E}_1^* = \langle x, \left(\frac{7}{10},\frac{5}{10},\frac{1}{10}\right),\left(\frac{5}{10},\frac{5}{10},\frac{4}{10}\right),\left(\frac{3}{10},\frac{5}{10},\frac{2}{10}\right)\rangle.$$

$$\mathcal{E}_2^* = \langle x, \left(\frac{1}{10},\frac{5}{10},\frac{2}{10}\right),\left(\frac{3}{10},\frac{5}{10},\frac{4}{10}\right),\left(\frac{4}{10},\frac{5}{10},\frac{6}{10}\right)\rangle.$$

Define a Neutrosophic function $\ddot{f}$: $(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}}) \longrightarrow (\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$ by $\ddot{f}$ (a) = b, $\ddot{f}$ (b) = c, $\ddot{f}$ (c) = a. $\mathcal{E}_2^*$ is a NOS in $(\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$. Also $\mathcal{E}_2^*$ is NαOS, NPOS, NSOS and $\mathcal{N}\beta OS$ in $\mathcal{Y}_{\mathcal{N}}$ and

$$\ddot{f}^{-1}(\mathcal{E}_2^*) = \langle x, \left(\frac{3}{10},\frac{5}{10},\frac{4}{10}\right),\left(\frac{4}{10},\frac{5}{10},\frac{6}{10}\right),\left(\frac{1}{10},\frac{5}{10},\frac{2}{10}\right)\rangle$$

Nint(Ncl($\ddot{f}^{-1}(\mathcal{E}_2^*)$)) = 1_N. Since $\ddot{f}^{-1}(\mathcal{E}_2^*)$ $\subseteq$ Nint(Ncl($\ddot{f}^{-1}(\mathcal{E}_2^*)$)) $\ddot{f}^{-1}(\mathcal{E}_2^*)$,is a NPOS in $(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}})$ and also $\ddot{f}^{-1}(\mathcal{E}_2^*)$ is $\mathcal{N}\beta OS$ in $\mathcal{X}_{\mathcal{N}}$. Thus $\ddot{f}$ is a Npre irresolute function, Npre-α-irresolute function, Npre continuous and $\mathcal{N}\beta$ -continuous. Now $\ddot{f}^{-1}(\mathcal{E}_2^*)$ $\nsubseteq$ Ncl(Nint($\ddot{f}^{-1}(\mathcal{E}_2^*)$)) = 0_N. So $\ddot{f}^{-1}(\mathcal{E}_2^*)$ is not NSOS in $\mathcal{X}_{\mathcal{N}}$. Also $\ddot{f}^{-1}(\mathcal{E}_2^*)$ $\nsubseteq$ Nint(Ncl(Ncl(($\ddot{f}^{-1}(\mathcal{E}_2^*)$))))= 0_N. Hence $\ddot{f}^{-1}(\mathcal{E}_2^*)$ is not NαOS in $\mathcal{X}_{\mathcal{N}}$.Thus $\ddot{f}$ is not Nα-irresolute function, not NS-α-irresolute function and not Nα-continuous and not NSemi continuous.

Example 3.17 Let $\mathcal{X}_{\mathcal{N}}$={a,b} $\mathcal{Y}_{\mathcal{N}}$={c,d} and $\mathcal{T}_{\mathcal{N}}$= {0, $\mathcal{E}_1^*$, 1}, $\Gamma_{\mathcal{N}}$= {0, $\mathcal{E}_2^*$, 1},are NTS on $\mathcal{X}_{\mathcal{N}}$ and $\mathcal{Y}_{\mathcal{N}}$ respectively where

$$\mathcal{E}_1^* = \langle x, \left(\frac{4}{10},\frac{5}{10},\frac{4}{10}\right),\left(\frac{5}{10},\frac{5}{10},\frac{2}{10}\right)\rangle$$

$$\mathcal{E}_2^* = \langle y, \left(\frac{4}{10},\frac{5}{10},\frac{5}{10}\right),\left(\frac{5}{10},\frac{5}{10},\frac{5}{10}\right)\rangle$$

$$\mathcal{E}_3^* = \langle y, \left(\frac{2}{10},\frac{5}{10},\frac{2}{10}\right),\left(\frac{4}{10},\frac{5}{10},\frac{5}{10}\right)\rangle$$

is a NS in $\mathcal{Y}_{\mathcal{N}}$. Define an Neutrosophic function $\ddot{f}$: $(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}}) \longrightarrow (\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$. by $\ddot{f}$(a) =c, $\ddot{f}$(b)=d. $\mathcal{E}_2^*$ is a NOS in $(\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$. Also $\mathcal{E}_2^*$ is NαOS, NPOS in $\mathcal{Y}_{\mathcal{N}}$.

$$\ddot{f}^{-1}(\mathcal{E}_2^*) = \langle x, \left(\frac{4}{10},\frac{5}{10},\frac{2}{10}\right),\left(\frac{3}{10},\frac{5}{10},\frac{5}{10}\right)\rangle$$

and$\ddot{f}^{-1}(\mathcal{E}_2^*)$ $\subseteq$ Nint(Ncl($\ddot{f}^{-1}(\mathcal{E}_2^*)$))= 1_N.

Thus $\ddot{f}^{-1}(\mathcal{E}_2^*)$ $\subseteq$ Nint(Ncl ($\ddot{f}^{-1}(\mathcal{E}_2^*)$)) Hence $\ddot{f}^{-1}(\mathcal{E}_2^*)$ is a NPOS in $\mathcal{X}_{\mathcal{N}}$.

Now $\mathcal{E}_3^* \subseteq$ Nint(Ncl($\mathcal{E}_3^*$) = 1_NTherefore $\mathcal{E}_3^*$ is an NPOS in $\mathcal{Y}_\mathcal{N}$. Also $\mathcal{E}_3^*$ is an NβOS in $\mathcal{Y}_\mathcal{N}$

$$\ddot{f}^{-1}(\mathcal{E}_3^*) = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle$$

Nint(Ncl($\ddot{f}^{-1}(\mathcal{E}_3^*)$)= 0_N.Thus $\ddot{f}^{-1}(\mathcal{E}_3^*) \subseteq$ Nint(Ncl($\ddot{f}^{-1}(\mathcal{E}_3^*)$)).

Hence $\ddot{f}^{-1}(\mathcal{E}_3^*)$ is not an NPOS in $\mathcal{X}_\mathcal{N}$.So $\ddot{f}$ is not Npre-β-irresolute function and $\ddot{f}$ is not Npre irresolute function. Since $\ddot{f}^{-1}(\mathcal{E}_3^*) \nsubseteq Ncl\left(\text{Nint}\left(\text{Ncl}(\mathcal{E}_3^*)\right)\right) = 0_N$.

$\ddot{f}^{-1}(\mathcal{E}_3^*)$is not N$\beta$OS in $\mathcal{X}_\mathcal{N}$. So $\ddot{f}$ is not Nβ-irresolute function.

Example 3.18 Let $\mathcal{X}_\mathcal{N}$={a,b} $\mathcal{Y}_\mathcal{N}$={c,d} and $\mathcal{T}_\mathcal{N}$= {0, $\mathcal{E}_1^*$, 1}, $\Gamma_\mathcal{N}$= {0, $\mathcal{E}_2^*$, 1},are NTS on $\mathcal{X}_\mathcal{N}$ and $\mathcal{Y}_\mathcal{N}$ respectively where

$$\mathcal{E}_1^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{2}{10}\right)\rangle$$

$$\mathcal{E}_2^* = \langle y, \left(\frac{4}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle$$

$$\mathcal{E}_3^* = \langle y, \left(\frac{2}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{2}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle$$

is a NS in $\mathcal{Y}_\mathcal{N}$. Define an Neutrosophic function $\ddot{f}$: ($\mathcal{X}_\mathcal{N}$,$\mathcal{T}_\mathcal{N}$) $\longrightarrow$ ($\mathcal{Y}_\mathcal{N}$,$\mathcal{G}_\mathcal{N}$) by $\ddot{f}$ (a) =c, $\ddot{f}$ (b)=d. $\mathcal{E}_2^*$ is a NOS in ($\mathcal{Y}_\mathcal{N}$,$\mathcal{G}_\mathcal{N}$). Also $\mathcal{E}_2^*$ is NαOS, NPOS in $\mathcal{Y}_\mathcal{N}$.

$$\ddot{f}^{-1}(\mathcal{E}_2^*) = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle$$

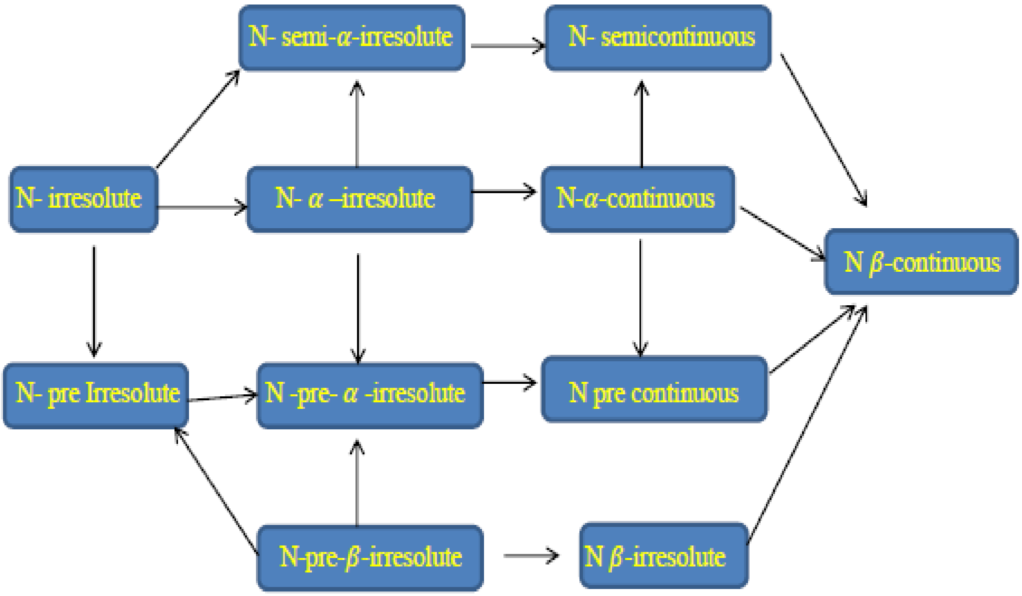
and Nint(Ncl $\ddot{f}^{-1}(\mathcal{E}_2^*)$) = 1_N .Thus $\ddot{f}^{-1}(\mathcal{E}_2^*) \subseteq$Nint(Ncl $\ddot{f}^{-1}(\mathcal{E}_2^*)$). Hence $\ddot{f}^{-1}(\mathcal{E}_2^*)$ is a NPOS in $\mathcal{X}_\mathcal{N}$.Therefore $\ddot{f}$ is a Npre irresolute, Npre-α-irresolute and Npre continuous.

$\mathcal{E}_3^*$ is a NS in $\mathcal{Y}_\mathcal{N}$ and $\mathcal{E}_3^* \subseteq$Ncl(Nint($\ddot{f}^{-1}(\mathcal{E}_3^*)$=$\overline{\mathcal{E}_2^*}$. Hence $\mathcal{E}_3^*$ is a $\mathcal{N}\beta OS$ in $\mathcal{Y}_\mathcal{N}$.

$$\ddot{f}^{-1}(\mathcal{E}_3^*) = \langle x, \left(\frac{2}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{2}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle$$

and Nint(Ncl($\ddot{f}^{-1}(\mathcal{E}_3^*)$) = 0_N.Thus $\ddot{f}^{-1}(\mathcal{E}_3^*) \nsubseteq$ Nint(Ncl($\ddot{f}^{-1}(\mathcal{E}_3^*)$). So $\ddot{f}^{-1}(\mathcal{E}_3^*)$ is not an NPOS in $\mathcal{X}_\mathcal{N}$. Hence $\ddot{f}$ is not Npre-β-irresolute function.

Diagram: I



4.PROPERTIES

Theorem 4.1 If a function $\ddot{f}$: $(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}}) \longrightarrow (\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$ is Npre-α-irresolute (Nα-irresolute and NS-α-irresolute, resp.) then $\ddot{f}^{-1}(\mathcal{E}_1^*)$ is $\mathcal{N}PCS$ (Nα-closed and NSemiclosed, resp.) in $\mathcal{X}_{\mathcal{N}}$ for any N nowhere dense set $\mathcal{E}_1^*$ of $\mathcal{Y}_{\mathcal{N}}$.

Proof:

Let $\mathcal{E}_1^*$ be an N nowhere dense set in $\mathcal{Y}_{\mathcal{N}}$. Then Nint(Ncl($\mathcal{E}_1^*$))= 0_N. Now, $\overline{\text{Nint(Ncl}(\mathcal{E}_1^*))} = 1_N \Rightarrow$ $\overline{\text{Ncl(Ncl}(\mathcal{E}_1^*))} = 1_N$ which implies Ncl(Nint($\overline{\mathcal{E}_1^*}$)) $= 1_N$. Since Nint 1_N $= 1_N$ .Hence $\overline{\mathcal{E}_1^*} \subseteq$ Nint(Ncl(Nint($\overline{\mathcal{E}_1^*}$) Then $\overline{\mathcal{E}_1^*}$is a N$\alpha$OS in $\mathcal{Y}_{\mathcal{N}}$. Since $\ddot{f}$ is Npre-α-irresolute (Nα-irresolute and N semi-α-irresolute, resp.), $\ddot{f}^{-1}(\overline{\mathcal{E}_1^*})$) is a NPOS (N$\alpha$OS and NSOS, resp.) in $\mathcal{X}_{\mathcal{N}}$.Hence $\ddot{f}^{-1}(\mathcal{E}_1^*)$) is a NPCS (N$\alpha$CS and NSCS, resp.) in $\mathcal{X}_{\mathcal{N}}$.

Theorem 4.2 If a function $\ddot{f}$: $(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}}) \longrightarrow (\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$ is Npre-β-irresolute, then $\ddot{f}^{-1}(\mathcal{E}_1^*)$ is $\mathcal{N}PCS$ in $\mathcal{X}_{\mathcal{N}}$ for any Nnowheredense set $\mathcal{E}_1^*$ of $\mathcal{Y}_{\mathcal{N}}$.

Proof: Let $\mathcal{E}_1^*$ be an Nnowhere dense set in $\mathcal{Y}_{\mathcal{N}}$. Then Nint(Ncl($\mathcal{E}_1^*$)) = 0_N. Now,$\overline{\text{Nint(Ncl}(\mathcal{E}_1^*))} = 1_N$. $\Rightarrow$ $\overline{\text{Ncl(Ncl}(\mathcal{E}_1^*))} = 1_N$ which implies Ncl(Nint($\overline{\mathcal{E}_1^*}$)) $= 1_N$ Since Nint1_N $= 1_N$and Ncl(Nint($\overline{\mathcal{E}_1^*}$) $\subseteq$ Ncl(Nint(Nint($\overline{\mathcal{E}_1^*}$)). Hence $\overline{\mathcal{E}_1^*}\subseteq$ 1_N =Ncl(Nint($\overline{\mathcal{E}_1^*}$) $\subseteq$Ncl(Nint(Ncl($\overline{\mathcal{E}_1^*}$)). Then $\overline{\mathcal{E}_1^*}$ is a $\mathcal{N}\beta OS$ in $\mathcal{Y}_{\mathcal{N}}$. Since $\ddot{f}$ is Npre-β-irresolute, $\ddot{f}^{-1}(\mathcal{E}_1^*)$is a NPOS in $\mathcal{X}_{\mathcal{N}}$. Hence $\ddot{f}^{-1}(\mathcal{E}_1^*)$ is a NPCS in $\mathcal{X}_{\mathcal{N}}$.

Theorem 4.3 A function $\ddot{f}$: $(\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}}) \longrightarrow (\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$ from an NTS $\mathcal{X}_{\mathcal{N}}$ into an NTS $\mathcal{Y}_{\mathcal{N}}$ is Npre-α-irresolute if and only if for each NP $\wp(\alpha,\beta)$) in $\mathcal{X}_{\mathcal{N}}$ and NαOS $\mathcal{E}_2^*$ in $\mathcal{Y}_{\mathcal{N}}$ such that $\ddot{f}$ ($\wp(\alpha,\beta)$) $\mathcal{E}_2^*$, there exists an NPOS $\mathcal{E}_1^*$ in $\mathcal{X}_{\mathcal{N}}$ such that $\wp(\alpha,\beta)$)$\in$ $\mathcal{E}_1^*$ and $\ddot{f}$ ($\mathcal{E}_1^*$)) $\subseteq$ $\mathcal{E}_2^*$.

Proof: Let $\ddot{f}$ be any Npre-α-irresolute function. $\wp(\alpha,\beta)$) be an NP in $\mathcal{X}_{\mathcal{N}}$ and $\mathcal{E}_2^*$ be any NαOS in $\mathcal{Y}_{\mathcal{N}}$ such that $\ddot{f}$ ($\wp(\alpha,\beta)$) $\in \mathcal{E}_2^*$. Then $\ddot{f}^{-1}(\mathcal{E}_2^*)$. Let $\mathcal{E}_1^*$=$\ddot{f}^{-1}(\mathcal{E}_2^*)$.Then $\mathcal{E}_1^*$ is a NPOS in $\mathcal{X}_{\mathcal{N}}$ which containing NP $\wp(\alpha,\beta)$) and $\ddot{f}$ ($\mathcal{E}_1^*$)))=$\ddot{f}^{-1}(\mathcal{E}_2^*)\subseteq\mathcal{E}_2^*$.Conversely, let $\mathcal{E}_2^*$ be a NαOS in $\mathcal{Y}_{\mathcal{N}}$ and $\wp(\alpha,\beta)$) be an NP in $\mathcal{X}_{\mathcal{N}}$ such that $\wp(\alpha,\beta) \in \ddot{f}^{-1}(\mathcal{E}_2^*)$. According to an assumption, there exists an NPOS $\mathcal{E}_1^*$ in $\mathcal{X}_{\mathcal{N}}$ such that $\wp(\alpha,\beta)$)$\in$ $\mathcal{E}_1^*$ and $\ddot{f}(\mathcal{E}_1^*)$)$\subseteq\mathcal{E}_2^*$. Hence $\wp(\alpha,\beta)$) $\in$ $\mathcal{E}_1^*$ $\subseteq$ $\ddot{f}^{-1}(\mathcal{E}_2^*)$. Also $\wp(\alpha,\beta)$) $\in$ $\mathcal{E}_1^*\subseteq$Nint((Ncl($\mathcal{E}_1^*$) $\subseteq$ Nint(Ncl$\ddot{f}^{-1}(\mathcal{E}_2^*)$. Therefore, $\ddot{f}^{-1}(\mathcal{E}_2^*)\subseteq$Nint(Ncl($\ddot{f}^{-1}(\mathcal{E}_2^*)$)is NPOS in $\mathcal{X}_{\mathcal{N}}$. Thus, $\ddot{f}$ is a Npre-α-irresolute function.

Theorem 4.4.A function $\ddot{f}: (\mathcal{X}_{\mathcal{N}}, \mathcal{T}_{\mathcal{N}}) \longrightarrow (\mathcal{Y}_{\mathcal{N}}, \mathcal{G}_{\mathcal{N}})$ from an NTS $\mathcal{X}_{\mathcal{N}}$ into an NTS $\mathcal{Y}_{\mathcal{N}}$ is -α-irresolute if and only if for each NP $p(\alpha,\beta)$ in $\mathcal{X}_{\mathcal{N}}$ and NαOS $\mathcal{E}_2^*$ in $\mathcal{Y}_{\mathcal{N}}$ such that $(p(\alpha,\beta))\ \mathcal{E}_2^*$, there exists an NαOS $\mathcal{E}_1^*$ in $\mathcal{X}_{\mathcal{N}}$ such that $p(\alpha,\beta))\in\ \mathcal{E}_1^*$ and $\ddot{f}\ (\mathcal{E}_1^*)) \subseteq \mathcal{E}_2^*$.

Proof: Let $\ddot{f}$ be any Nα-irresolute function. $p(\alpha,\beta))$ be an NP in $\mathcal{X}_{\mathcal{N}}$ and $\mathcal{E}_2^*$ be any NαOS in $\mathcal{Y}_{\mathcal{N}}$ such that $\ddot{f}\ (p(\alpha,\beta))\in\ \mathcal{E}_2^*$. Then $p(\alpha,\beta)\ \in \ddot{f}^{-1}(\mathcal{E}_2^*)$ =Nαint$\ddot{f}^{-1}(\mathcal{E}_2^*)$.Let $\mathcal{E}_1^*$= Nαint$\ddot{f}^{-1}(\mathcal{E}_2^*)$.Then $\mathcal{E}_1^*$ is a NαOS in $\mathcal{X}_{\mathcal{N}}$ which containing NP $p(\alpha,\beta))$ and $\ddot{f}(\mathcal{E}_1^*) = \ddot{f}($Nαint$\ddot{f}^{-1}(\mathcal{E}_2^*) = f(\ddot{f}^{-1}(\mathcal{E}_2^*)) \subseteq \mathcal{E}_2^*$ .Conversely, let $\mathcal{E}_2^*$ be an NαOS in $\mathcal{Y}_{\mathcal{N}}$ and $p(\alpha,\beta))$ be an NP in $\mathcal{X}_{\mathcal{N}}$ such that $p(\alpha,\beta))\in\ \ddot{f}^{-1}(\mathcal{E}_2^*)$. According to an assumption, there exists an NαOS $\in\ \mathcal{E}_1^*$ in $\mathcal{X}_{\mathcal{N}}$ such that $p(\alpha,\beta))\in\mathcal{E}_1^*$ and $\ddot{f}$ $(\mathcal{E}_1^*)\subseteq\ \mathcal{E}_2^*$.Hence $p(\alpha,\beta))\in\ \mathcal{E}_1^*\ \subseteq\ \ddot{f}^{-1}(\mathcal{E}_2^*))$ and $p(\alpha,\beta))\in\ \mathcal{E}_1^*\ =\ddot{f}^{-1}(\mathcal{E}_2^*)=$ Nαint$A \subseteq$ Nαint$\ddot{f}^{-1}(\mathcal{E}_2^*)$. Since $p(\alpha,\beta))$ be an arbitrary NP and $\ddot{f}^{-1}(\mathcal{E}_2^*)$ is union of all NPs containing in $\ddot{f}^{-1}(\mathcal{E}_2^*)$,which gives that $\ddot{f}^{-1}(\mathcal{E}_2^*)\subseteq$ Nint$\ddot{f}^{-1}(\mathcal{E}_2^*)$is NαOS in $\mathcal{X}_{\mathcal{N}}$. Hence $\ddot{f}$ is a Nα-irresolute function.

Theorem 4.5 $\mathcal{E}_1^*$ function $\ddot{f}: (\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}}) \longrightarrow (\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$ from an NTS $\mathcal{X}_{\mathcal{N}}$ into an NTS $\mathcal{Y}_{\mathcal{N}}$ is N semi-α-irresolute if and only if for each NP $p(\alpha,\beta))$ in $\mathcal{X}_{\mathcal{N}}$ and NαOS $\mathcal{E}_2^*$ in $\mathcal{Y}_{\mathcal{N}}$ such that $\ddot{f}$ $(p(\alpha,\beta))\in \mathcal{E}_2^*$, there exists an NSOS $\mathcal{E}_1^*$ in $\mathcal{X}_{\mathcal{N}}$ such that $p(\alpha,\beta))\in \mathcal{E}_1^*$ and $\ddot{f}\ (\mathcal{E}_1^*)) \subseteq \mathcal{E}_2^*$.

Proof: Let $\ddot{f}$ be any NS-α-irresolute function, $p(\alpha,\beta))$ be an NP in $\mathcal{X}_{\mathcal{N}}$ and $\mathcal{E}_2^*$ be any NαOS in $\mathcal{Y}_{\mathcal{N}}$ such that $\ddot{f}\ (p(\alpha,\beta))\in \mathcal{E}_2^*$. Then $p(\alpha,\beta)\ \in (\mathcal{E}_2^*)$. Let $\mathcal{E}_1^*=\ddot{f}^{-1}(\mathcal{E}_2^*)$.Then $\mathcal{E}_1^*$ is a NSOS in $\mathcal{X}_{\mathcal{N}}$ which containing NP $p(\alpha,\beta))$ and $\ddot{f}\ (\mathcal{E}_1^*)=\ddot{f}(\ \ddot{f}^{-1}(\mathcal{E}_2^*))\subseteq\ \mathcal{E}_2^*$

Conversely, let $\mathcal{E}_2^*$ be an NαOS in $\mathcal{Y}_{\mathcal{N}}$ and $p(\alpha,\beta))$ be an NP in $\mathcal{X}_{\mathcal{N}}$ such that $p(\alpha,\beta)\in \ddot{f}^{-1}(\mathcal{E}_2^*)$. According to an assumption, there exists an NSOS $\mathcal{E}_1^*$ in $\mathcal{X}_{\mathcal{N}}$ such that $p(\alpha,\beta))\ \mathcal{E}_1^*$ and $\ddot{f}\ (\mathcal{E}_1^*)) \subseteq \mathcal{E}_2^*$.Hence $p(\alpha,\beta))\in \mathcal{E}_1^*\ \subseteq\ \ddot{f}^{-1}(\mathcal{E}_2^*)$. Also $p(\alpha,\beta)\ \in\mathcal{E}_1^*\ \subseteq$ Ncl(Nint($\mathcal{E}_1^*$) $\subseteq$ Ncl(Nint($\ddot{f}^{-1}(\mathcal{E}_2^*)$)Therefore, $\ddot{f}^{-1}(\mathcal{E}_2^*)\subseteq$ Ncl(Nint($\ddot{f}^{-1}(\mathcal{E}_2^*)$)is NSOS in $\mathcal{X}_{\mathcal{N}}$. Hence $\ddot{f}$ is a NS-α-irresolute function

Theorem 4.6 A function $\ddot{f}: (\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}}) \longrightarrow (\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$ from an NTS $\mathcal{X}_{\mathcal{N}}$ into an NTS $\mathcal{Y}_{\mathcal{N}}$ is N pre-β-irresolute if and only if for each NP $p(\alpha,\beta)$ in $\mathcal{X}_{\mathcal{N}}$ and $\mathcal{N}\beta\mathcal{OS}$ $\mathcal{E}_2^*$ in $\mathcal{Y}_{\mathcal{N}}$ such that $\ddot{f}$ $(p(\alpha,\beta))\in\mathcal{E}_2^*$, there exists an NPOS $\mathcal{E}_1^*$ in $\mathcal{X}_{\mathcal{N}}$ such that $p(\alpha,\beta))\in \mathcal{E}_1^*$ and $\ddot{f}\ (\mathcal{E}_1^*)) \subseteq \mathcal{E}_2^*$.

Proof: Let $\ddot{f}$ be any Npre-β-irresolute mapping. $p(\alpha,\beta))$ be an NP in $\mathcal{X}_{\mathcal{N}}$ and $\mathcal{E}_2^*$ be any $\mathcal{N}\beta\mathcal{OS}$ in $\mathcal{Y}_{\mathcal{N}}$ such that $\ddot{f}\ (p(\alpha,\beta))\in \mathcal{E}_2^*$. Then $(p(\alpha,\beta))\in \ddot{f}^{-1}(\mathcal{E}_2^*)$. Let $\mathcal{E}_1^*=\ddot{f}^{-1}(\mathcal{E}_2^*)$.Then $\mathcal{E}_1^*$ is a NPOS in $\mathcal{X}_{\mathcal{N}}$ which containing NP $p(\alpha,\beta))$ and $\ddot{f}\ (\mathcal{E}_1^*)))\ f(\ddot{f}^{-1}(\mathcal{E}_2^*))\subseteq\ \mathcal{E}_2^*$.

Conversely, let $\mathcal{E}_2^*$ be an $\mathcal{N}\beta\mathcal{OS}$ in $\mathcal{Y}_{\mathcal{N}}$ and $p(\alpha,\beta))$ be an NP in $\mathcal{X}_{\mathcal{N}}$ such that $p(\alpha,\beta))\in \ddot{f}^{-1}(\mathcal{E}_2^*)$. According to an assumption, there exists an NPOS $\mathcal{E}_1^*$ in $\mathcal{X}_{\mathcal{N}}$ such that $p(\alpha,\beta))\in\ \mathcal{E}_1^*$ and $\ddot{f}\ (\mathcal{E}_1^*)\subseteq$ $\mathcal{E}_2^*$. Hence $p(\alpha,\beta)\ \in\mathcal{E}_1^*\ \subseteq\ \ddot{f}^{-1}(\mathcal{E}_2^*)$.Also $p(\alpha,\beta)\ \in\mathcal{E}_1^*\ \subseteq$ Nint((Ncl($\mathcal{E}_1^*$)) $\subseteq$ Nint(Ncl$\ddot{f}^{-1}(\mathcal{E}_2^*)$). Therefore, $\ddot{f}^{-1}(\mathcal{E}_2^*)\ \subseteq$ Nint(Ncl$\ddot{f}^{-1}(\mathcal{E}_2^*)$). is NPOS in $\mathcal{X}_{\mathcal{N}}$. Hence $\ddot{f}$ is a Npre-β-irresolute function.

Theorem 4.7 A function $\ddot{f}: (\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}}) \longrightarrow (\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$ from an NTS $\mathcal{X}_{\mathcal{N}}$ into an NTS $\mathcal{Y}_{\mathcal{N}}$ is N pre-α-irresolute if and only if for each NP $p(\alpha,\beta))$ in $\mathcal{X}_{\mathcal{N}}$ and NαOS $\mathcal{E}_2^*$ in $\mathcal{Y}_{\mathcal{N}}$ such that $\ddot{f}$ $(p(\alpha,\beta))\in \mathcal{E}_2^*$,Ncl($\ddot{f}^{-1}(\mathcal{E}_2^*)$)is a NN of NP $p(\alpha,\beta))$ in $\mathcal{X}_{\mathcal{N}}$.

Proof: Let $\ddot{f}$ be any Npre-α-irresolute function. $p(\alpha,\beta))$ be an NP in $\mathcal{X}_{\mathcal{N}}$ and $\mathcal{E}_2^*$ be any NαOS in $\mathcal{Y}_{\mathcal{N}}$ such that $\ddot{f}\ (p(\alpha,\beta))\in \mathcal{E}_2^*$. Then $p(\alpha,\beta))\in \ddot{f}^{-1}(\mathcal{E}_2^*)\subseteq$Nint(Ncl ($\ddot{f}^{-1}(\mathcal{E}_2^*)$)). Hence Ncl($\ddot{f}^{-1}(\mathcal{E}_2^*)$) is IFN of $p(\alpha,\beta))$ in $\mathcal{X}_{\mathcal{N}}$.

Conversely, let $\mathcal{E}_2^*$ be a NαOS in $\mathcal{Y}_{\mathcal{N}}$ and $p(\alpha,\beta))$ be an NP in $\mathcal{X}_{\mathcal{N}}$ such that $\ddot{f}\ (p(\alpha,\beta))\in \mathcal{E}_2^*$. Then $p(\alpha,\beta))\in\ \ddot{f}^{-1}(\mathcal{E}_2^*)$ According to an assumption,*Ncl(*$\ddot{f}^{-1}(\mathcal{E}_2^*)$)is NN of NP $p(\alpha,\beta))$ in $\mathcal{X}_{\mathcal{N}}$. So $p(\alpha,\beta))\in$Nint(Ncl($\ddot{f}^{-1}(\mathcal{E}_2^*)$). Thus $\ddot{f}^{-1}(\mathcal{E}_2^*)\subseteq$ Nint(Ncl $\ddot{f}^{-1}(\mathcal{E}_2^*)$). Hence $\ddot{f}^{-1}(\mathcal{E}_2^*)$is a NPOS in $\mathcal{X}_{\mathcal{N}}$. Therefore $\ddot{f}$ is a Npre-α-irresolute function.

Theorem 4.8:

A function $\ddot{f}: (\mathcal{X}_{\mathcal{N}},\mathcal{T}_{\mathcal{N}}) \longrightarrow (\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$ from an NTS $\mathcal{X}_{\mathcal{N}}$ into an NTS $\mathcal{Y}_{\mathcal{N}}$ is N pre-β-irresolute if and only if for each NP $p(\alpha,\beta))$ in $\mathcal{X}_{\mathcal{N}}$ and $\mathcal{N}\beta OS$ $\mathcal{E}_2^*$ in $\mathcal{Y}_{\mathcal{N}}$ such that $\ddot{f}$ $p(\alpha,\beta))\in \mathcal{E}_2^*$,N*cl*($\ddot{f}^{-1}(\mathcal{E}_2^*)$), is a NN of NP $p(\alpha,\beta))$ in $\mathcal{X}_{\mathcal{N}}$.

Proof: Let $\ddot{f}$ be any Npre-β-irresolute function. $p(\alpha,\beta))$ be an NP in $\mathcal{X}_{\mathcal{N}}$ and $\mathcal{E}_2^*$ be any $\mathcal{N}\beta OS$ in $\mathcal{Y}_{\mathcal{N}}$ such that $\ddot{f}$ $(p(\alpha,\beta))\in \mathcal{E}_2^*$. Then $p(\alpha,\beta))\in \ddot{f}^{-1}(\mathcal{E}_2^*)\subseteq$Nint(Ncl($\ddot{f}^{-1}(\mathcal{E}_2^*)\subseteq$ Ncl(Nint($\ddot{f}^{-1}(\mathcal{E}_2^*)$). Hence Ncl(Nint$\ddot{f}^{-1}(\mathcal{E}_2^*)$) is IFN of $p(\alpha,\beta))$ in $\mathcal{X}_{\mathcal{N}}$.

Conversely, let $\mathcal{E}_2^*$ be an $\mathcal{N}\beta OS$ in $\mathcal{Y}_{\mathcal{N}}$ and $p(\alpha,\beta))$ be an NP in $\mathcal{X}_{\mathcal{N}}$ such that $\ddot{f}(p(\alpha,\beta))\in \mathcal{E}_2^*$. Then $p(\alpha,\beta))\in \ddot{f}^{-1}(\mathcal{E}_2^*)$ According to an assumption,N*cl(Nint* $\ddot{f}^{-1}(\mathcal{E}_2^*)$) is nN of NP$p(\alpha,\beta))$ in $\mathcal{X}_{\mathcal{N}}$. Thus $p(\alpha,\beta))\in$ Nint(Ncl($\ddot{f}^{-1}(\mathcal{E}_2^*)$), so $\ddot{f}^{-1}(\mathcal{E}_2^*)\subseteq$Nint(Ncl($\ddot{f}^{-1}(\mathcal{E}_2^*)$). Hence $\ddot{f}^{-1}(\mathcal{E}_2^*)$is a NPOS in $\mathcal{X}_{\mathcal{N}}$. Therefore $\ddot{f}$ is a Npre-β-irresolute function.

Theorem 4.9

The following hold for functions $\ddot{f}$: $\mathcal{X}_{\mathcal{N}} \longrightarrow \mathcal{Y}_{\mathcal{N}}$ and $\ddot{g}$: $\mathcal{Y}_{\mathcal{N}} \longrightarrow \mathcal{Z}_{\mathcal{N}}$

i) If $\ddot{f}$ is Npre irresolute and g is Npre-α-irresolute (Npre-β-irresolute, resp.), then $\ddot{g} \circ \ddot{f}$ is N pre-α-irresolute (Npre-β-irresolute, resp.) function.

ii) If $\ddot{f}$ is Npre-α- irresolute (Npre-β-irresolute, resp.), and g is Nα-continuous ($\mathcal{N}\beta$ − continuous, resp.), then $\ddot{g} \circ \ddot{f}$ is N pre continuous.

iii) If $\ddot{f}$ is Npre-α-irresolute (Npre-β-irresolute, resp.) and g is Nα-irresolute ($\mathcal{N}\beta$ − irresolute, resp.), then $\ddot{g} \circ \ddot{f}$ is Npre-α-irresolute (Npre-β-irresolute, resp.).

iv) If $\ddot{f}$ is NS-α-irresolute (Nα-irresolute, resp.) and g is IFα-continuous, then $\ddot{g} \circ \ddot{f}$ is N Semi continuous (Nα-continuous, resp.).

v) If $\ddot{f}$ is NS-α-irresolute (Nα-irresolute, resp.) and g is IFα-irresolute, then $\ddot{g} \circ \ddot{f}$ is NS-α-irresolute (Nα-irresolute, resp.).

vi) If $\ddot{f}$ is Nirresolute and g is NS-α-irresolute, then $\ddot{g} \circ \ddot{f}$ is NS-α-irresolute.

vii) If $\ddot{f}$ is Nα-irresolute and g is Nstrongly α-continuous, then $\ddot{g} \circ \ddot{f}$ is Nstrongly α-continuous.

Proof:

(i) Let $\mathcal{E}_2^*$ be an NαOS ($\mathcal{N}\beta OS$, resp.) in Z. Since g is Npre-α-irresolute (Npre-β-irresolute, resp.) $\ddot{g}^{-1}(\mathcal{E}_2^*)$is a NPOS in $\mathcal{Y}_{\mathcal{N}}$.Now $(\ddot{g} \circ \ddot{f})^{-1}(\mathcal{E}_2^*) = \ddot{f}^{-1}(\ddot{g}^{-1}(\mathcal{E}_2^*))$. Since $\ddot{f}$ is Npre irresolute,$\ddot{f}^{-1}(\ddot{g}^{-1}(\mathcal{E}_2^*))$ is a NPOS in $\mathcal{X}_{\mathcal{N}}$. Hence $\ddot{g} \circ \ddot{f}$ is Npre-α-irresolute (Npre-β-irresolute, resp.).

(ii) Let $\mathcal{E}_2^*$ be an NOS in Z. Since g is Nα-continuous ($\mathcal{N}\beta$ -continuous, resp.), $\ddot{g}^{-1}(\mathcal{E}_2^*)$ is a NαOS ($\mathcal{N}\beta OS$, resp.) in $\mathcal{Y}_{\mathcal{N}}$.Now $(\ddot{g} \circ \ddot{f})^{-1}(\mathcal{E}_2^*)= \ddot{f}^{-1}(\ddot{g}^{-1}(\mathcal{E}_2^*))$. Since $\ddot{f}$ is Npre-α-irresolute (Npre-β-irresolute, resp.), $\ddot{f}^{-1}(\ddot{g}^{-1}(\mathcal{E}_2^*))$is a NPOS in $\mathcal{X}_{\mathcal{N}}$. Hence $\ddot{g} \circ \ddot{f}$ is Npre continuous.

(iii) Let $\mathcal{E}_2^*$ be an NαOS ($\mathcal{N}\beta OS$, resp.) in Z. Since g is Nα-irresolute ($\mathcal{N}\beta$ -irresolute, resp.), $\ddot{g}^{-1}(\mathcal{E}_2^*)$ is a NαOS ($\mathcal{N}\beta OS$ resp.) in $\mathcal{Y}_{\mathcal{N}}$. Now $(\ddot{g} \circ \ddot{f})^{-1}(\mathcal{E}_2^*)= \ddot{f}^{-1}(\ddot{g}^{-1}(\mathcal{E}_2^*))$. Since $\ddot{f}$ is Npre-α-irresolute (Npre-β-irresolute, resp.), $\ddot{f}^{-1}(\ddot{g}^{-1}(\mathcal{E}_2^*))$ is a NPOS in $\mathcal{X}_{\mathcal{N}}$. Hence $\ddot{g} \circ \ddot{f}$ is Npre-α-irresolute (Npre-β-irresolute, resp.).

(iv) Let $\mathcal{E}_2^*$ be an NOS in Z. Since g is Nα-continuous, $\ddot{g}^{-1}(\mathcal{E}_2^*)$ is an NαOS in $\mathcal{Y}_{\mathcal{N}}$. Now $(\ddot{g} \circ \ddot{f})^{-1}(\mathcal{E}_2^*)$ $\ddot{f}^{-1}(\ddot{g}^{-1}(\mathcal{E}_2^*))$. Since $\ddot{f}$ isNS-α-irresolute (Nα-irresolute, resp.), $\ddot{f}^{-1}(\ddot{g}^{-1}(\mathcal{E}_2^*))$ is a NSOS (NαOS, resp.) in $\mathcal{X}_{\mathcal{N}}$. Hence $\ddot{g} \circ \ddot{f}$ is NSemi continuous (Nα-continuous, resp.).

(v) Let $\mathcal{E}_2^*$ be an NαOS in Z. Since g is Nα-irresolute, $\ddot{g}^{-1}(\mathcal{E}_2^*)$ is an NαOS in $\mathcal{Y}_{\mathcal{N}}$. Now Since $\ddot{f}$ is NS-α-irresolute (Nα- irresolute, resp.), $\ddot{f}^{-1}(\ddot{g}^{-1}(\mathcal{E}_2^*))$ is a NSOS (NαOS, resp.) in $\mathcal{X}_{\mathcal{N}}$. Hence $\ddot{g} \circ \ddot{f}$ is NS-α-irresolute (Nα-irresolute, resp.).

(vi) Let $\mathcal{E}_2^*$ be an NαOS in Z. Since g is NS-α-irresolute, $\ddot{g}^{-1}(\mathcal{E}_2^*)$ is a NSOS in $\mathcal{Y}_{\mathcal{N}}$. Now $(\ddot{g} \circ \ddot{f})^{-1}(\mathcal{E}_2^*)= \ddot{f}^{-1}(\ddot{g}^{-1}(\mathcal{E}_2^*))$. Since $\ddot{f}$ is Nirresolute, $\ddot{f}^{-1}(\ddot{g}^{-1}(\mathcal{E}_2^*))$is a NSOS in $\mathcal{X}_{\mathcal{N}}$. Hence $\ddot{g} \circ \ddot{f}$ is NS-α-irresolute.

(vii) Let $\mathcal{E}_2^*$ be an NSOS in Z. Since g is Nstrongly α-continuous $\ddot{g}^{-1}(\mathcal{E}_2^*)$ is a NαOS in $\mathcal{Y}_{\mathcal{N}}$. Now $(\ddot{g} \circ \ddot{f})^{-1}(\mathcal{E}_2^*)$ $\ddot{f}^{-1}(\ddot{g}^{-1}(\mathcal{E}_2^*))$. Since $\ddot{f}$ is Nα-irresolute, $\ddot{f}^{-1}(\ddot{g}^{-1}(\mathcal{E}_2^*))$ is a NαOS in $\mathcal{X}_{\mathcal{N}}$. Hence $\ddot{g} \circ \ddot{f}$ is Nstrongly α-continuous.

5 . CHARACTERIZATIONS

In this section, several characterizations of Neutrosophic pre-α-irresolute functions, Neutrosophic α-irresolute functions, Neutrosophic semi-α-irresolute functions and Neutrosophic pre-β-irresolute functions are established

Theorem 5.1 If $\ddot{f}$ is a function from an NTS $(\mathcal{X}_{\mathcal{N}},\mathcal{J}_{\mathcal{N}})$ to another NTS $(\mathcal{Y}_{\mathcal{N}},\mathcal{G}_{\mathcal{N}})$, then the following are equivalent.

(a) $\ddot{f}$ is a Npre-α-irresolute.

(b) $\ddot{f}^{-1}(\mathcal{E}_2^*) \subseteq \text{int}\,(cl\,(\ddot{f}^{-1}(\mathcal{E}_2^*)))$ for every NαOS $\mathcal{E}_2^*$ in $\mathcal{Y}_{\mathcal{N}}$.

(c) $\ddot{f}^{-1}(\mathcal{E}_3^*)$ is $\mathcal{NPCS}$ in $\mathcal{X}_{\mathcal{N}}$ for every NαCS $\mathcal{E}_3^*$ in $\mathcal{Y}_{\mathcal{N}}$.

(d) Ncl (Nint $(\ddot{f}^{-1}(\mathcal{E}_4^*))) \subseteq \ddot{f}^{-1}(N\alpha cl(\mathcal{E}_4^*)$ for every NS $\mathcal{E}_4^*$ of $\mathcal{Y}_{\mathcal{N}}$.

(e) $\ddot{f}$ (Ncl (Nint $\mathcal{E}_5^*$)) ⊆N$\alpha clf\,(\mathcal{E}_5^*)$ for every NS $\mathcal{E}_5^*$ of $\mathcal{X}_{\mathcal{N}}$.

Proof:

(a)⇒ (b): Let $\mathcal{E}_2^*$ be an NαOS in $\mathcal{Y}_{\mathcal{N}}$. By (a), $\ddot{f}^{-1}(\mathcal{E}_2^*)$ is $\mathcal{NPOS}$ in $\mathcal{X}_{\mathcal{N}}$. $\ddot{f}^{-1}(\mathcal{E}_2^*) \subseteq$ Nint(Ncl($\ddot{f}^{-1}(\mathcal{E}_2^*)$)). Hence (a)⇒ (b) is proved.

(b)⇒ (c): Let $\mathcal{E}_3^*$ be any NαCS in $\mathcal{Y}_{\mathcal{N}}$. Then $\overline{\mathcal{E}_3^*}$ is NαOS in $\mathcal{Y}_{\mathcal{N}}$. By (b),$(\ddot{f}^{-1}(\mathcal{E}_3^*)\ \overline{\mathcal{E}_3^*})$ ⊆Nint(Ncl $\ddot{f}^{-1}(\overline{\mathcal{E}_3^*})$). But $\overline{\ddot{f}^{-1}(\mathcal{E}_3^*)} \subseteq$ Nint(Ncl($\overline{\ddot{f}^{-1}(\mathcal{E}_3^*)}$)))= Nint(Nint($\overline{\ddot{f}^{-1}(\mathcal{E}_3^*)}$)))= $\overline{Ncl(N\imath nt(\ddot{f}^{-1}(\mathcal{E}_3^*))}$

This implies $\overline{\ddot{f}^{-1}(\mathcal{E}_3^*)} \subseteq \overline{Ncl(N\imath nt(\ddot{f}^{-1}(\mathcal{E}_3^*))}$ $\Rightarrow Ncl(Nint(\ddot{f}^{-1}(\mathcal{E}_3^*) \subseteq \ddot{f}^{-1}(\mathcal{E}_3^*) \Rightarrow \ddot{f}^{-1}(\mathcal{E}_3^*)$ is $\mathcal{NPCS}$ in $\mathcal{X}_{\mathcal{N}}$. Hence(b) ⇒ (c) is proved.

(c) ⇒ (d): Let $\mathcal{E}_4^*$ be an $\mathcal{NS}$ in $\mathcal{Y}_{\mathcal{N}}$. Then Nαcl($\mathcal{E}_4^*$)) is Nα-closed in $\mathcal{Y}_{\mathcal{N}}$. ⇒ $\ddot{f}^{-1}$(Nαcl($\mathcal{E}_4^*$))) is $\mathcal{NPCS}$ in $\mathcal{X}_{\mathcal{N}}$. Then Ncl(Nint($\ddot{f}^{-1}$(Nαcl($\mathcal{E}_4^*$)))⊆(Nαcl($\mathcal{E}_4^*$))) Hence (c)⇒ (d) is proved.

(d)⇒ (e): Let $\mathcal{E}_5^*$ be an NS in $\mathcal{X}_{\mathcal{N}}$. Then Ncl(Nint($\mathcal{E}_5^*$) ⊆ Ncl(Nint$\ddot{f}^{-1}\big(\ddot{f}(\mathcal{E}_5^*)\big)$ ⊆ Ncl(Nint($\ddot{f}^{-1}\big($Nαcl$\ddot{f}(\mathcal{E}_5^*)\big)$)⊆ Nαcl$\ddot{f}(\mathcal{E}_5^*)$⊆ Then Ncl(Nint(($\mathcal{E}_5^*$)) ⊆$\ddot{f}^{-1}\big($Nαcl$\ddot{f}(\mathcal{E}_5^*)\big)$).

Thus $\ddot{f}(Ncl(Nint((\mathcal{E}_5^*))\subseteq$Nαcl$\ddot{f}(\mathcal{E}_5^*)$). Hence (d) ⇒ (e) isproved.

(e)⇒ (a): Let $\mathcal{E}_2^*$ be an NαOS in $\mathcal{Y}_{\mathcal{N}}$. Then $\ddot{f}^{-1}(\mathcal{E}_2^*) = \overline{\ddot{f}^{-1}(\mathcal{E}_2^*)}$ = Nint(Ncl($\ddot{f}^{-1}(\mathcal{E}_2^*)$))is a NS in $\mathcal{X}_{\mathcal{N}}$. By (e),

$$\ddot{f}(Ncl(Nint(\ddot{f}^{-1}(\mathcal{E}_2^*))) \subseteq Nacl(\ddot{f}\left(\ddot{f}^{-1}(\mathcal{E}_2^*)\right) \subseteq Nacl\left(\overline{\mathcal{E}_2^*}\right) = \overline{Na\imath nt(\mathcal{E}_2^*)} = \overline{\mathcal{E}_2^*}$$

Thus, $\ddot{f}(Ncl(Nint(\ddot{f}^{-1}(\mathcal{E}_2^*))) \subseteq Nacl(\ddot{f}\left(\ddot{f}^{-1}(\mathcal{E}_2^*)\right)\subseteq \overline{\mathcal{E}_2^*}$. ------(1)

Consider

$\overline{N\imath nt(Ncl(\ddot{f}^{-1}(\mathcal{E}_2^*))} = Ncl\overline{(Ncl(\ddot{f}^{-1}(\mathcal{E}_2^*))}$ =Ncl(Nint$\overline{(\ddot{f}^{-1}(\mathcal{E}_2^*))} \subseteq \ddot{f}^{-1}\ddot{f}(Ncl\left(Nint(\ddot{f}^{-1}(\overline{\mathcal{E}_2^*}))\right)$------(2)

By (1) and (2), $\overline{N\imath nt(Ncl(\ddot{f}^{-1}(\mathcal{E}_2^*))} \subseteq \ddot{f}^{-1}\ddot{f}(Ncl\left(Nint(\ddot{f}^{-1}(\overline{\mathcal{E}_2^*}))\right))$ ⊆Nint(Ncl($\ddot{f}^{-1}(\mathcal{E}_2^*)$)) ⇒ $\ddot{f}^{-1}(\overline{\mathcal{E}_2^*}) = \overline{\ddot{f}^{-1}(\mathcal{E}_2^*)} \Rightarrow \ddot{f}^{-1}(\mathcal{E}_2^*) \subseteq$ Nint((Ncl($\ddot{f}^{-1}(\mathcal{E}_2^*)$))⇒ $\ddot{f}^{-1}(\mathcal{E}_2^*)is\mathcal{N}POS$ in $\mathcal{X}_\mathcal{N}$. Thus $\ddot{f}$ is Npre-α-irresolute. Hence (e)⇒ (a) is proved.

Theorem 5.2 If $\ddot{f}: (\mathcal{X}_\mathcal{N},\mathcal{T}_\mathcal{N}) \longrightarrow (\mathcal{Y}_\mathcal{N},\mathcal{G}_\mathcal{N})$ be a mapping from NTS $\mathcal{X}_\mathcal{N}$ into NTS $\mathcal{Y}_\mathcal{N}$.
Then the following are equivalent.

(a) $\ddot{f}$ is Nα-irresolute.

(b) $\ddot{f}^{-1}(\mathcal{E}_2^*)$ is NαCS in $\mathcal{X}_\mathcal{N}$ for each NαCS $\mathcal{E}_2^*$ in $\mathcal{Y}_\mathcal{N}$.

(c) $\ddot{f}$ (Nα*clA*) ⊆Nα*clf* ($\mathcal{E}_1^*$))) for each NS $\mathcal{E}_1^*$ in $\mathcal{X}_\mathcal{N}$.

(d) Nα*cl* ($\ddot{f}^{-1}$ ($\mathcal{E}_2^*$)) ⊆ $\ddot{f}^{-1}$(Nα*cl*($\mathcal{E}_2^*$)for each NS $\mathcal{E}_2^*$ in $\mathcal{Y}_\mathcal{N}$.

(e) $\ddot{f}^{-1}$(Nα int $\mathcal{E}_2^*$) ⊆Nα int $\ddot{f}^{-1}(\mathcal{E}_2^*)$ *for* each NS $\mathcal{E}_2^*$ in $\mathcal{Y}_\mathcal{N}$.

Proof:

(a)⇒(b): Let $\mathcal{E}_2^*$ be NαCS in $\mathcal{Y}_\mathcal{N}$. Then $\overline{\mathcal{E}_2^*}$ is NαOS in $\mathcal{Y}_\mathcal{N}$. Since $\ddot{f}$ is Nα-irresolute, $\ddot{f}^{-1}(\mathcal{E}_2^*) = \overline{\ddot{f}^{-1}(\mathcal{E}_2^*)}$is N$\alpha$OS in $\mathcal{X}_\mathcal{N}$. Hence $\ddot{f}^{-1}(\mathcal{E}_2^*)$ is NαCS in $\mathcal{X}_\mathcal{N}$.Thus (a)⇒(b) is proved.

(b)⇒(c): Let $\mathcal{E}_1^*$ be NS in $\mathcal{X}_\mathcal{N}$. Then $\mathcal{E}_1^* \subseteq \ddot{f}^{-1}(\ddot{f}(\mathcal{E}_1^*)) \subseteq \ddot{f}^{-1}(N\alpha cl(\ddot{f}(\mathcal{E}_1^*))$ .As $N\alpha cl(\ddot{f}(\mathcal{E}_1^*))$ is NαCS in $\mathcal{Y}_\mathcal{N}$, by(b), $\ddot{f}^{-1}(N\alpha cl(\ddot{f}(\mathcal{E}_1^*))$ is a NαCS in $\mathcal{X}_\mathcal{N}$. $N\alpha cl(\mathcal{E}_1^*) \subseteq N\alpha cl((N\alpha cl\left(\ddot{f}(\mathcal{E}_1^*)\right) = N\alpha cl\left(\ddot{f}(\mathcal{E}_1^*)\right)$Hence (b)⇒(c) is proved.

(c)⇒(d): For any NS $\mathcal{E}_2^*$ in $\mathcal{Y}_\mathcal{N}$, let $\ddot{f}^{-1}(\mathcal{E}_2^*) = \mathcal{E}_1^*$ By (c), $\ddot{f}(N\alpha cl\left(\ddot{f}^{-1}(\mathcal{E}_2^*)\right) \subseteq N\alpha cl\ddot{f}\left(\ddot{f}^{-1}(\mathcal{E}_2^*)\right) \subseteq N\alpha cl(\mathcal{E}_2^*)$.and $\alpha cl\left(\ddot{f}^{-1}(\mathcal{E}_2^*)\right) \subseteq \ddot{f}^{-1}\left(\ddot{f}(N\alpha cl\left(\ddot{f}^{-1}(\mathcal{E}_2^*)\right)\right) \subseteq \ddot{f}^{-1}\left(\ddot{f}(N\alpha cl\left(\ddot{f}^{-1}(\mathcal{E}_2^*)\right)\right) \subseteq \ddot{f}^{-1}(N\alpha cl(\mathcal{E}_2^*))$

Thus $N\alpha cl\ddot{f}^{-1}(\mathcal{E}_2^*) \subseteq \ddot{f}^{-1}(N\alpha cl(\mathcal{E}_2^*))$ Hence (c)⇒(d) is proved.

(d)⇒(e): For any NS $\mathcal{E}_2^*$ in $\mathcal{Y}_\mathcal{N}$, $N\alpha int(\mathcal{E}_2^*) = \overline{N\alpha cl(\overline{\mathcal{E}_2^*})}$.Now$\ddot{f}^{-1}(N\alpha int(\mathcal{E}_2^*)) = \ddot{f}^{-1}\left(\overline{N\alpha cl(\overline{\mathcal{E}_2^*})}\right) = \overline{\ddot{f}^{-1}\left(N\alpha cl(\overline{\mathcal{E}_2^*})\right)}$= $\overline{N\alpha cl\ddot{f}^{-1}(\overline{\mathcal{E}_2^*})} = \overline{N\alpha\imath nt\overline{\ddot{f}^{-1}(\mathcal{E}_2^*)}} \subseteq N\alpha int\ddot{f}^{-1}(\mathcal{E}_2^*)$

(e)⇒(a): Let $\mathcal{E}_2^*$ be NαOS in $\mathcal{Y}_\mathcal{N}$. Then $\mathcal{E}_2^* = N\alpha int(\mathcal{E}_2^*)$ and $\ddot{f}^{-1}(\mathcal{E}_2^*) = \ddot{f}^{-1}(\mathcal{E}_2^*) \subseteq N\alpha int(\mathcal{E}_2^*)$. By definition $\ddot{f}^{-1}(\mathcal{E}_2^*) \supseteq N\alpha int$ ($\ddot{f}^{-1}(\mathcal{E}_2^*)$).So $\ddot{f}^{-1}(\mathcal{E}_2^*)$=$N\alpha int$ ($\ddot{f}^{-1}(\mathcal{E}_2^*)$).Thus $\ddot{f}^{-1}(\mathcal{E}_2^*)$ is a NαOS in $\mathcal{X}_\mathcal{N}$ which implies $\ddot{f}$ is Nα-irresolute. Thus (e)⇒(a) is proved.

Theorem 5.3 If $\ddot{f}$ is a function from an $\mathcal{NTS}(\mathcal{X}_\mathcal{N},\mathcal{T}_\mathcal{N})$ to another NTS $(\mathcal{Y}_\mathcal{N},\mathcal{G}_\mathcal{N})$,
then the following are equivalent.

(a) $\ddot{f}$ is a NS-α-irresolute.

(b) $\ddot{f}^{-1}(\mathcal{E}_2^*)$ ⊆ N*cl* (Nint ($\ddot{f}^{-1}$ ($\mathcal{E}_2^*$))) for every NαOS $\mathcal{E}_2^*$ in $\mathcal{Y}_\mathcal{N}$.

(c) $\ddot{f}^{-1}$ ($\mathcal{E}_3^*$) is NSemiclosed in $\mathcal{X}_\mathcal{N}$ for every Nα-closed set $\mathcal{E}_3^*$ in $\mathcal{Y}_\mathcal{N}$.

(d) Nint (N*cl* ($\ddot{f}^{-1}(\mathcal{E}_4^*)$)) ⊆ $\ddot{f}^{-1}$(Nα*cl* $\mathcal{E}_4^*$) for every NS $\mathcal{E}_4^*$ of $\mathcal{Y}_\mathcal{N}$.

(e) $\ddot{f}$ (Nint (N*clE*)) ⊆Nα*clf* ($\mathcal{E}_5^*$)for every NS $\mathcal{E}_5^*$ of $\mathcal{X}_\mathcal{N}$.

Proof: (a)⇒ (b):

Let $\mathcal{E}_2^*$ be an NαOS in $\mathcal{Y}_\mathcal{N}$. By (a), $\ddot{f}^{-1}(\mathcal{E}_2^*)$ is $\mathcal{N}SOS$ in $\mathcal{X}_\mathcal{N}$. ⇒ $\ddot{f}^{-1}(\mathcal{E}_2^*)$ ⊆Ncl(Nint $\ddot{f}^{-1}(\mathcal{E}_2^*)$. Hence (a)⇒ (b) is proved.

(b)⇒ (c): Let $\mathcal{E}_3^*$ be any NαCS in $\mathcal{Y}_{\mathcal{N}}$. Then $\overline{\mathcal{E}_3^*}$ is a NαOS in $\mathcal{Y}_{\mathcal{N}}$. By (b), $\ddot{f}^{-1}(\overline{\mathcal{E}_3^*}) \subseteq$ Ncl(Nint($\ddot{f}^{-1}(\overline{\mathcal{E}_3^*})$. But $\overline{\ddot{f}^{-1}(\mathcal{E}_3^*)} \subseteq$ Ncl(Nint($\ddot{f}^{-1}(\overline{\mathcal{E}_3^*})$) = $\overline{\text{Ncl(Ncl}(\ddot{f}^{-1}(\mathcal{E}_3^*))}$ = Nint(Ncl($\ddot{f}^{-1}(\mathcal{E}_3^*)$) ⊆ $\ddot{f}^{-1}(\mathcal{E}_3^*) \Rightarrow \ddot{f}^{-1}(\mathcal{E}_3^*)$ is NSemiclosed in $\mathcal{X}_{\mathcal{N}}$. Hence (b)⇒ (c) is proved.

(c)⇒ (d): Let $\mathcal{E}_4^*$ be an NS in $\mathcal{Y}_{\mathcal{N}}$. Then Nα-cl($\mathcal{E}_4^*$) is Nα-closed in $\mathcal{Y}_{\mathcal{N}}$. By (c) $\ddot{f}^{-1}$(Nαcl($\mathcal{E}_4^*$))) is NSemiclosed in $\mathcal{X}_{\mathcal{N}}$. Then Nint(Ncl$\ddot{f}^{-1}$(Nαcl($\mathcal{E}_4^*$))) ⊆$\ddot{f}^{-1}$(Nαcl($\mathcal{E}_4^*$))). Hence (c)⇒ (d) is proved.

(d)⇒ (e): Let $\mathcal{E}_5^*$ be an NS in $\mathcal{X}_{\mathcal{N}}$. Then Nint(Ncl($\mathcal{E}_5^*$) ⊆ Nint(Ncl$\ddot{f}^{-1}(\ddot{f}(\mathcal{E}_5^*$)))⊆ Nint(Ncl$\ddot{f}^{-1}(N\alpha cl\ddot{f}(\mathcal{E}_5^*$)))Therefore Nint(Ncl($\mathcal{E}_5^*$)⊆ $\ddot{f}^{-1}(N\alpha cl\ddot{f}(\mathcal{E}_5^*$))).

Consequently $\ddot{f}(Nint(Ncl(\mathcal{E}_5^*) \subseteq N\alpha cl\ddot{f}(\mathcal{E}_5^*)$. Hence (d) ⇒(e) is proved.

(e)⇒ (a): Let $\mathcal{E}_2^*$ be an NαOS in $\mathcal{Y}_{\mathcal{N}}$. Then $\ddot{f}^{-1}(\overline{\mathcal{E}_2^*})$= $\overline{\ddot{f}^{-1}(\mathcal{E}_2^*)}$))is a NS in $\mathcal{X}_{\mathcal{N}}$.

By (e), $\ddot{f}\big(Nint(Ncl(\ddot{f}^{-1}(\overline{\mathcal{E}_2^*})\big) \subseteq N\alpha cl\ddot{f}(\ddot{f}^{-1}(\overline{\mathcal{E}_2^*})) \subseteq N\alpha cl(\overline{\mathcal{E}_2^*}) = \overline{N\alpha int(\mathcal{E}_2^*)} = \overline{\mathcal{E}_2^*}$

Thus, $\ddot{f}\big(Nint(Ncl(\ddot{f}^{-1}(\overline{\mathcal{E}_2^*})\big) \subseteq \overline{\mathcal{E}_2^*}$. ------(1)

Consider $Ncl(Nint\left(\ddot{f}^{-1}(\mathcal{E}_2^*)\right) = Nint(Nint\left(\ddot{f}^{-1}(\mathcal{E}_2^*)\right) = Nint(Ncl\left(\ddot{f}^{-1}(\overline{\mathcal{E}_2^*})\right)$

$Nint(Ncl(\ddot{f}^{-1}(\mathcal{E}_2^*)) \subseteq \ddot{f}^{-1}(\ddot{f}\big(Nint(Ncl(\ddot{f}^{-1}(\mathcal{E}_2^*)))\big)$)-----(2)

By (1) and (2),

$\overline{Ncl(Nint\left(\ddot{f}^{-1}(\mathcal{E}_2^*)\right)} \subseteq \ddot{f}^{-1}(\ddot{f}\big(Nint(Ncl(\ddot{f}^{-1}(\mathcal{E}_2^*)))\big)$)

$\subseteq \ddot{f}^{-1}(\overline{\mathcal{E}_2^*}) = \overline{\ddot{f}^{-1}(\overline{\mathcal{E}_2^*})} \Rightarrow \ddot{f}^{-1}(\mathcal{E}_2^*) \subseteq Ncl(Nint(\ddot{f}^{-1}(\mathcal{E}_2^*)) \Rightarrow \ddot{f}^{-1}(\mathcal{E}_2^*)$ is$\mathcal{N}SOS$ in $\mathcal{X}_{\mathcal{N}}$.

Thus $\ddot{f}$ is NS-α-irresolute. Hence (e)⇒ (a) is proved.

Theorem 5.4 If $\ddot{f}$ is a function from an NTS ($\mathcal{X}_{\mathcal{N}}$,$\mathcal{T}_{\mathcal{N}}$) to another NTS ($\mathcal{Y}_{\mathcal{N}}$,$\mathcal{G}_{\mathcal{N}}$), then the following are equivalent.

(a) $\ddot{f}$ is a Npre-β-irresolute.

(b) $\ddot{f}^{-1}(\mathcal{E}_2^*)$ ⊆ int (*cl* ($\ddot{f}^{-1}$ ($\mathcal{E}_2^*$))) for every $\mathcal{N}\beta OS$ $\mathcal{E}_2^*$ in $\mathcal{Y}_{\mathcal{N}}$.

(c) $\ddot{f}^{-1}$ ($\mathcal{E}_3^*$) is $\mathcal{N}PCS$ in $\mathcal{X}_{\mathcal{N}}$ for every $\mathcal{N}\beta$ -closed set $\mathcal{E}_3^*$ in $\mathcal{Y}_{\mathcal{N}}$.

(d) *cl* (int ($\ddot{f}^{-1}(\mathcal{E}_4^*)$)) ⊆ $\ddot{f}^{-1}(N\beta clD)$for every NS $\mathcal{E}_4^*$ of $\mathcal{Y}_{\mathcal{N}}$.

(e) $\ddot{f}$ (*cl* (int $\mathcal{E}_5^*$)) ⊆ $N\beta cl\ddot{f}$ ($\mathcal{E}_5^*$)for every NS $\mathcal{E}_5^*$ of $\mathcal{X}_{\mathcal{N}}$.

Proof: (a)⇒ (b): Let $\mathcal{E}_2^*$ be an $\mathcal{N}\beta OS$ in $\mathcal{Y}_{\mathcal{N}}$. By (a), $\ddot{f}^{-1}(\mathcal{E}_2^*)$ is $\mathcal{N}POS$ in $\mathcal{X}_{\mathcal{N}}$ ⇒ $\ddot{f}^{-1}(\mathcal{E}_2^*)$⊆ $Nint(Ncl(\ddot{f}^{-1}(\mathcal{E}_2^*))$. Hence (a)⇒ (b) is proved.

(b)⇒ (c): Let $\mathcal{E}_3^*$ be any $\mathcal{N}\beta CS$ in $\mathcal{Y}_{\mathcal{N}}$. Then $\overline{\mathcal{E}_3^*}$ is $\mathcal{N}\beta OS$ in $\mathcal{Y}_{\mathcal{N}}$. By (b), $\ddot{f}^{-1}(\overline{\mathcal{E}_3^*})$⊆ $Nint(Ncl(\ddot{f}^{-1}(\overline{\mathcal{E}_3^*}))$. But $\overline{\ddot{f}^{-1}(\mathcal{E}_3^*)}$⊆ Nint(N(cl($\overline{\ddot{f}^{-1}(\mathcal{E}_3^*)}$) = Nint(N(int($\overline{\ddot{f}^{-1}(\mathcal{E}_3^*)}$) =$\overline{\text{Ncl(Nınt}(\ddot{f}^{-1}(\mathcal{E}_3^*)))}$ ⇒ $\overline{\ddot{f}^{-1}(\mathcal{E}_3^*)}$⊆$\overline{\text{Ncl(Nınt}(\ddot{f}^{-1}(\mathcal{E}_3^*)))}$.

This implies Ncl(Nint($\ddot{f}^{-1}(\mathcal{E}_3^*)$)) ⊆ $\ddot{f}^{-1}(\mathcal{E}_3^*) \Rightarrow \ddot{f}^{-1}(\mathcal{E}_3^*)$is $\mathcal{N}PCS$ in $\mathcal{X}_{\mathcal{N}}$.

Hence (b) ⇒ (c) is proved.

(c)⇒ (d): Let $\mathcal{E}_4^*$ be an NS in $\mathcal{Y}_{\mathcal{N}}$. Then $N\beta cl(\mathcal{E}_4^*)$ is $\mathcal{N}\beta$ -closed in $\mathcal{Y}_{\mathcal{N}}$. By (c),$\overline{\ddot{f}^{-1}(N\beta cl(\mathcal{E}_4^*)}$)is $\mathcal{N}PCS$ in $\mathcal{X}_{\mathcal{N}}$. Then ($\ddot{f}^{-1}(N\beta cl(\mathcal{E}_4^*))$⊆ $\ddot{f}^{-1}(N\beta cl(\mathcal{E}_4^*))$

Thus Ncl(Nint($\overline{\ddot{f}^{-1}(N\beta cl(\mathcal{E}_4^*)}$)⊆ $\ddot{f}^{-1}(N\beta cl(\mathcal{E}_4^*))$

Hence (c) ⇒ (d) is proved.

(d)⇒ (e):

Let $\mathcal{E}_5^*$ be an NS in $\mathcal{X}_{\mathcal{N}}$. Then Ncl(Nint($\mathcal{E}_5^*$)) ⊆ Ncl(Nint$\ddot{f}^{-1}(\ddot{f}(\mathcal{E}_5^*$))) ⊆ Ncl(Nint($\ddot{f}^{-1}(N\beta cl\big(\ddot{f}(\mathcal{E}_5^*)\big))\big)$ ⊆ $\ddot{f}^{-1}(N\beta cl\big(\ddot{f}(\mathcal{E}_5^*)\big))\big)$ then Ncl(Nint($\mathcal{E}_5^*$)) ⊆$\ddot{f}^{-1}N\beta cl\big(\ddot{f}(\mathcal{E}_5^*)\big))\big)$.This implies $\ddot{f}(Ncl(Nint(\mathcal{E}_5^*))) \subseteq N\beta cl\big(\ddot{f}(\mathcal{E}_5^*)\big))\big)$. Hence (d)⇒ (e) is proved.

(e)⇒ (a)**:** Let $\mathcal{E}_2^*$ be an $\mathcal{N}\beta OS$ in $\mathcal{Y}_{\mathcal{N}}$. Then $\ddot{f}^{-1}$ ($\overline{\mathcal{E}_2^*}$)= $\overline{\ddot{f}^{-1}}$ ($\mathcal{E}_2^*$)is a NS in $\mathcal{X}_{\mathcal{N}}$.

By (e), $\ddot{f}(Ncl(Nint\left(\ddot{f}^{-1}\left(\overline{\mathcal{E}_2^*}\right)\right)) \subseteq N\beta cl\ddot{f}(\ddot{f}^{-1}((\overline{\mathcal{E}_2^*})) \subseteq \overline{\mathcal{E}_2^*}$------(1)Consider $\overline{\text{Nint}(\text{Ncl}(\ddot{f}^{-1}(\mathcal{E}_2^*))}$

$= \overline{\text{Ncl}(\text{Ncl}(\ddot{f}^{-1}(\mathcal{E}_2^*))}= \text{Nint}(\text{Ncl}(\ddot{f}^{-1}\,(\overline{\mathcal{E}_2^*})) \subseteq\ Ncl(Nint\left(\ddot{f}^{-1}\left(\overline{\mathcal{E}_2^*}\right)\right) \subseteq \ddot{f}^{-1}(\ddot{f}(Ncl(Nint(\ddot{f}^{-1}(\overline{\mathcal{E}_2^*}))$------(2)

By (1) and (2),$\overline{\text{Nint}(\text{Ncl}(\ddot{f}^{-1}(\mathcal{E}_2^*))} \subseteq\ \ddot{f}^{-1}(\ddot{f}(Ncl(Nint(\ddot{f}^{-1}(\overline{\mathcal{E}_2^*})) \subseteq \ddot{f}^{-1}\ (\overline{\mathcal{E}_2^*})= \overline{\ddot{f}^{-1}\,(\mathcal{E}_2^*)}$ This implies $\ddot{f}^{-1}\ (\mathcal{E}_2^*)= \ \subseteq \text{Nint}((\text{Ncl}(\ddot{f}^{-1}(\mathcal{E}_2^*))$ which proves $\ddot{f}^{-1}(\mathcal{E}_2^*)is\mathcal{N}POS$ in $\mathcal{X}_{\mathcal{N}}$. Thus $\ddot{f}$ is pre-β-irresolute. Hence (e)⇒ (a) is proved.

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On Measures of Similarity for Neutrosophic Sets with Applications in Classification and Evaluation Processes

Mahima Poonia[1] and Rakesh Kumar Bajaj [2, *]

[1] Jaypee University of Information Technology, Waknaghat; mahimapoonia1508@gmail.com
[2] Jaypee University of Information Technology, Waknaghat; rakesh.bajaj@juit.ac.in

* Correspondence: rakesh.bajaj@juit.ac.in; Tel.: (+91 – 7018926342)

Abstract: In the last decades, different researchers have considerably incorporated the notion of neutrosophic sets, their properties and different measures for managing the uncertainty, impreciseness and vagueness in the information. It may be noted that neutrosophic set is a popular defined procedures for solving the classification problem and evaluation problem of decision-making. Numerical examples for the classification problem and the decision-making problem have also been presented and compared the obtained results with the well established existing approaches.

Keywords: Neutrosophic set, Similarity measure, Classification problem, Decision-making

1. Introduction

In the fields of expert system, information & belief system, the concept of belongingness of fuzzy set (FS) [5] does not remain the single key-term to be taken care for the evident but also the non-belongingness grade to be taken into consideration. The intuitionistic fuzzy sets (IFSs) [6] take belongingness/non-belongingness both into account to manage the incomplete/imprecise information other than the indeterminate information of a belief system (if any). The technical literature of FSs and IFSs have been utilized in many real-world applications in the field of decision-making, pattern recognition problems, financial economics etc.

The concept of a neutrosophic set (NS) was first given by Smarandache [7] as an additional generalization for mathematically model the uncertainty/impreciseness, incompleteness/ inconsistency found in the problems. As in the words of Smarandache - "*Neutrosophy is a branch of philosophy which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra*"[7]. It may certainly be noted that the notion of neutrosophic set can be taken as a formalized general structure of the crisp set, FS, IFS etc. Single valued neutrosophic set (SVNS) is a particular case of neutrosophic sets explained by Wang et al. [4]. In the available research, different extensions of SVNSs are found having a composed structure of soft neutrosophic set, rough neutrosophic, hesitant NS etc. Many researchers have enhanced the literature of FS and IFS by studying various information measures of similarity, entropy, divergence etc. as having different applications in various fields. It is to mention that the indeterminacy degree of IFS is dependent on the membership & non-membership grade. In this way, a decision maker is bounded and restricted for quantifying the sense of impreciseness. The theory of neutrosophic set certainly have the capability to deal with such restrictions and proved to be effective in information-based applications.

The generalization of fuzzy set to neutrosophic set may be well understood by the geometric presentation in Figure 1 showing the better coverage of the imprecise information.

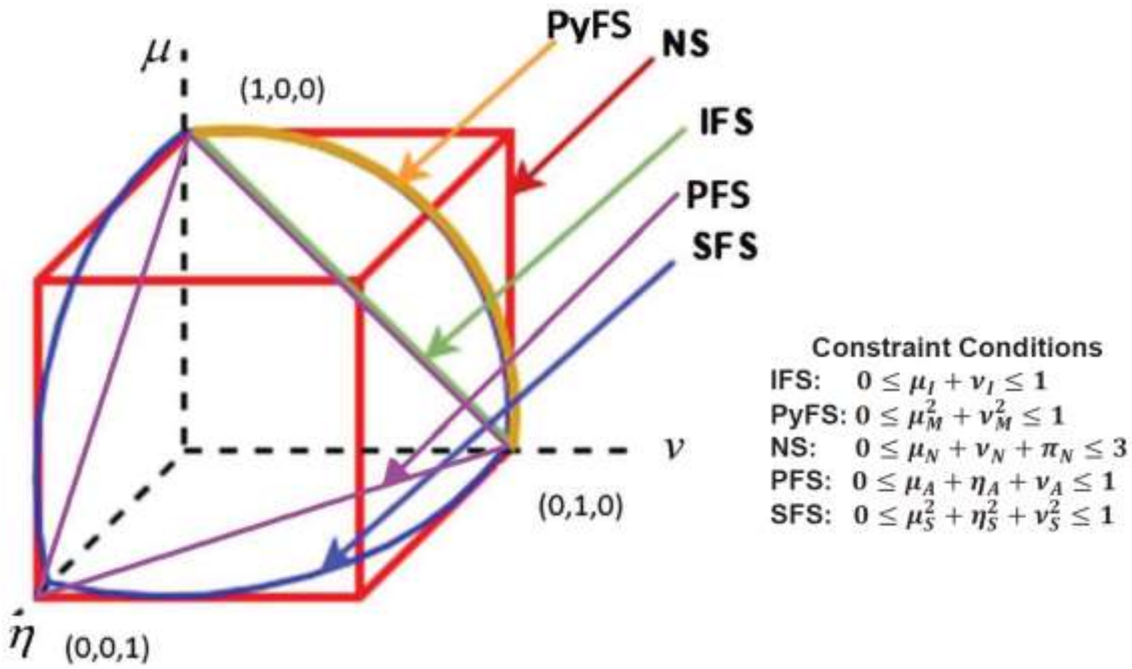

Figure 1: Extension of Fuzzy Set to Neutrosophic set

A brief literature survey on measures of neutrosophic sets is given below:

"Different kinds of similarity/distance measures of NSs have been well studied by Broumi and Smarandache [8]. Utilizing the distance measure between two SVNSs, Majumdar and Samanta [9] defined some important measures of similarity along with their characteristics. Ye [28] presented the three different similarity measures between SVNSs as an extension of the Jaccard, Dice, and cosine similarity measures in vector space and utilized then to solve the MCDM problem under simplified neutrosophic information. Mondal and Pramanik [29] proposed a new trigonometric measure called tangent similarity measure as an improvement of cosine similarity and used this to solve the applications problem of "selection of educational stream" and "medical diagnosis". Ye [10] has given different similarity measures for the interval neutrosophic sets based on distance measures with application in decision processes [11]. Next, Ye et al. [12] [13 and Wu et al. [15] discussed the problem of diagnosis based on the similarity measures for SVNSs.."

"Also, a new multi-attribute decision making method has been developed based on the proposed information measures with a numerical example of city pollution evaluation. Thao and Smarandache [16] proposed new divergence measure for neutrosophic set with some properties and utilized to solve the medical diagnosis problem and the classification problem. Recently, the notion of NSs theory and its various generalizations have been explored in various field of research by different researchers. Abdel-Basset et al. [17] developed a new model to handle the hospital medical care evaluation system based on plithogenic sets and also studied intelligent medical decision support model [18] based on soft computing and internet of things. In addition to this, a hybrid plithogenic approach [19] by utilizing the quality function in the supply chain management has also been developed. Further, a new systematic framework for providing aid and support to the cancer patients by using neutrosophic sets has been successfully suggested by Abdel-Basset et al. [20]. Based on neutrosophic sets, some

new decision-making models have also been successfully presented for project selection [21] and heart disease diagnosis [22] with advantages and defined limitations. In subsequent research, Abdel-Basset et al. [23] have proposed a modified forecasting model based on neutrosophic time series analysis and a new model for linear fractional programming based on triangular neutrosophic numbers [24]. Also, Yang et al. [25] have studied some new similarity and entropy measures of the interval neutrosophic sets on the basis of new axiomatic definition along with its application in MCDM problem."

Recently, Abdel-Basset et al. [30] proposed an integrated plithogenic MCDM approach for financial performance evaluation of manufacturing industries. Additionally, a novel decision-making model has been provided for sustainable supply chain finance under uncertainty environment [31]. Also, a novel framework to evaluate innovation value proposition for smart product–service systems has been well developed by Abdel-Basset et al. [32]. Guleria et al. [26] proposed a parametric divergence measure and along with it presented some methodologies for solving the classification problem and MCDM problem in neutrosophic set up. Guleria and Bajaj [27] provided a new technique for the dimensionality reduction of informational data under the neutrosophic soft matrices environment and utilized it to get the solution for the decision-making problem. Looking at the recent literature discussions accomplished above, it is being stated that the information measures of NSs deal with the concerns in connection with uncertainty/vagueness.

Nabeeh et al. [36] proposed the technique of N-MCDMF which integrates the theory of neutrosophic sets using different methods of MCDM for evaluating the GCP in the direction of environment. Further, Nabeeh et al. [37] [38] also enhanced the process of the management of the resources and clearly explained the internet of things connection in case of smart village by using the method of neutrosophic AHP and TOPSIS which helps the decision makers to solve the problem of reaching the goal of the companies respectively. Additionally, various examples have been presented which make the readers understand the utility of the used methods more accurately. The problem faced by the IoT industries was further explained by Basset et al. [39] and presented the solution to the traditional process using the non-traditional method in contrast with the methods of AHP and theory of neutrosophic.

In this paper, we have incorporated the exponential function for framing the new similarity measures for the neutrosophic sets along with their weighted form and utilized them for the solving a standard classification problem of pattern recognition and the decision-making problem. Various other researchers have also discussed various types of similarity measures

The structure of the presented manuscript is as follows:

Some fundamental definitions, standard operations and existing similarity measures of the neutrosophic sets are presented in Section 2. In section 3, we have proposed some new exponential similarity measures with proof of their validity and also presented several counter-intuitive cases to show the efficacy of the exponential measures. In order to show the applicability of the exponential similarity measures, we have presented the two illustrative examples - one related to the classification problem (pattern recognition) and other related to the evaluation problem of decision-making in Section 4. In addition, some important comparative remarks have been enumerated. Finally, we have concluded the paper in Section 5.

2. Preliminaries

First, we present soe basic preliminaries and fundamenta definitons in connection with neutrosophic set, similarity measures and its properties which are available in literature.

Definition 1 *[5]* "An intuitionistic fuzzy set (IFS) I in U (universe of discourse) is given by

$$I = \{< u, \mu_1(u), \nu_1(u) > | u \in U\};$$

Where $\mu_1 : U \to [0,1]$ and $\nu_1 : U \to [0,1]$ degree of membership and non-membership respectively and for every $u \in U$ satisfies the condition

$$0 \leq \mu_1(u) + \nu_1(u) \leq 1;$$

And the degree of indeterminacy for any IFS I and $u \in U$ is given by $\pi_1 = 1 - \mu_1(u) - \nu_1(u)$".

Definition 2 [7] "Let U be a fixed class points (objects) with a generic element u in U. A neutrosophic set P in U is specified by a truth-membership function $T_P(u)$, an indeterminacy-membership function $I_P(u)$ and falsity-membership function $F_P(u)$, where $T_P(u)$, $I_P(u)$ and $F_P(u)$ are real standard or nonstandard subsets of the interval $(^-0, 1^+)$ such that $T_P(u): U \to (^-0, 1^+), I_P(u): U \to (^-0, 1^+), F_P(u): U \to (^-0, 1^+)$ and the sum of these function viz. $T_P(u) + I_P(u) + F_P(u)$ satisfies the requirement

$$^-0 \leq supT_P(u) + supI_P(u) + supF_P(u) \leq 3^+."$$

We denote the neutrosophic set $I = \{(u, T_P(u),\ I_P(u),\ F_P(u) | u \in U\}$.

"In case of neutrosophic set, indeterminacy gets quantified in an explicit way, while truth-membership, indeterminacy-membership and falsity-membership are independent terms. Such framework is found to be very useful in the applications of information fusion where the data are logged from different sources. For scientific and engineering applications, Wang et al. [4] defined a single valued neutrosophic set (SVNS) as an instance of a neutrosophic set as follows:"

Definition 3 [4] "Let U be a fixed class of points (objects) with a generic element u in U. A single valued neutrosophic set P in U is characterzied by a truth-membership function $T_P(u)$, an interminacy-membership function $I_P(u)$ and a falsity-membership function $F_P(u)$. Foe each point $u \in U$, where $I_P(u), T_P(u), F_P(u) \in [0,1]$. A single valued neutrosophic set P can be denoted by

$$P = \{< I_P(u), T_P(u), F_P(u) | u \in U\}."$$

It may be noted that $I_P(u) + T_P(u) + F_P(u) \in [0,3]$.

We denote *SVNS(U)* as the collection of all the *SVNSs* on U. For any two single valued neutrosophic sets $P, Q \epsilon SVNS(U)$ (Refer [4]):

- **Union of P and Q:**

$$P \cup Q = \{(u, T_{P\cup Q}(u), I_{P\cup Q}(u), F_{P\cup Q}(u) | u \in U\};$$

where $T_{P\cup Q}(u) = max\{T_P(u), T_Q(u)\}$, $I_{P\cup Q}(u) = min\{I_P(u), I_Q(u)\}$ and

$F_{P\cup Q}(u) = min\{F_P(u), F_Q(u)\}$; for all $u \in U$

- **Intersection of P and Q:**

$$P \cap Q = \{(u, T_{P\cup Q}(u), I_{P\cup Q}(u), F_{P\cup Q}(u) | u \in U\};$$

where $T_{P\cap Q}(u) = min\{T_P(u), T_Q(u)\}$, $I_{P\cap Q}(u) = max\{I_P(u), I_Q(u)\}$ and $F_{P\cap Q}(u) = max\{F_P(u), F_Q(u)\}$; for all $u \in U$

- **Containment:**

$P \subseteq Q$ if and only if $T_P(u) \leq T_Q(u)$, $I_P(u) \geq I_Q(u), F_P(u) \geq F_Q(u)$, for all $u \in U$.

- **Complement:** The complement of P, denoted by $\overline{P}$, characterized by

$T_P(u) = 1 - T_P(u), T_P(u) = 1 - T_P(u)$, $T_P(u) = 1 - T_P(u),$; for all $u \in U$.

Definition 4 "A function $S: SVNS(U) \times SVNS(U) \Rightarrow [0,1]$ is called a similarity measure for single value neutrosophic sets, if the following conditions are satisfied:

For any $P, Q, O \in SVNS(U)$,

I. $0 \leq S(P,Q) \leq 1$

II. $S(P,Q) = 1$ if and only if $P = Q$;

III. $S(P,Q) = S(Q,P)$;

IV. $P \subseteq Q \subseteq O$, then $S(P,O) \leq S(P,Q); S(P,O) \leq S(Q,O)$."

Existing Similarity Measures

In the literature, different similarity measures have been proposed by various researchers. For the sake of understanding, some of them are being presented below.

Let $P = \{T_P(u_i), I_P(u_i), F_P(u_i) | u_i \in U\}$ & $Q = \{T_Q(u_i), I_Q(u_i), F_Q(u_i) | u_i \in U, i = 1,2, \dots n\}$ be the two-single valued neutrosophic sets. Then the existing similarity measures between P and Q are as follows:

- **Jaccard's Similarity Measure** [28]

$$S_j(P,Q) = \frac{1}{n}\sum_{i=1}^{n} \frac{T_P(u_i)T_Q(u_i)+I_P(u_i)I_Q(u_i)+F_P(u_i)F_Q(u_i)}{T_P^2(u_i)+I_P^2(u_i)+F_P^2(u_i)+T_Q^2(u_i)+I_Q^2(u_i)+F_Q^2(u_i)-\left(T_P(u_i)T_Q(u_i)+I_P(u_i)I_Q(u_i)+F_P(u_i)F_Q(u_i)\right)} \quad (2.1)$$

- **Dice Similarity Measure** [28]

$$S_D(P,Q) = \frac{1}{n}\sum_{i=1}^{n} \frac{2(T_P(u_i)T_Q(u_i)+I_P(u_i)I_Q(u_i)+F_P(u_i)F_Q(u_i))}{T_P^2(u_i)+I_P^2(u_i)+F_P^2(u_i)+T_Q^2(u_i)+I_Q^2(u_i)+F_Q^2(u_i)} \quad (2.2)$$

- **Cosine Similarity Measure** [28]

$$S_C(P,Q) = \frac{1}{n}\sum_{i=1}^{n} \frac{T_P(u_i)T_Q(u_i)+I_P(u_i)I_Q(u_i)+F_P(u_i)F_Q(u_i)}{\sqrt{T_P^2(u_i)+I_P^2(u_i)+F_P^2(u_i)}\sqrt{T_Q^2(u_i)+I_Q^2(u_i)+F_Q^2(u_i)}} \quad (2.3)$$

- **Tangent Similarity Measure** [29]

$$S_T(P,Q) = 1 - \frac{1}{n}\sum_{i=1}^{n} \tan\left(\frac{\pi}{12}\left(|T_P(u_i) - T_Q(u_i)| + |I_P(u_i) - I_Q(u_i)| + |F_P(u_i) - F_Q(u_i)|\right)\right). \quad (2.4)$$

- The similarity measure of SVNSs between P and Q is defined as follows [9]:

$$S_1 = \frac{\sum_{i=1}^{n}\left(min\left(T_P(u_i), T_Q(u_i)\right) + min\left(I_P(u_i), I_Q(u_i)\right) + min\left(F_P(u_i), F_Q(u_i)\right)\right)}{\sum_{i=1}^{n}\left(max\left(T_P(u_i), T_Q(u_i)\right) + max\left(I_P(u_i), I_Q(u_i)\right) + max\left(F_P(u_i), F_Q(u_i)\right)\right)} \quad (2.5)$$

- Similarity Measures Based on Theoretic Approach.[40]

$$S_{2T}(P,Q) = \frac{1}{n}\left(\sum_{1}^{N}\left[\frac{min\left(T_P(u_i), T_Q(u_i)\right)}{max\left(T_P(u_i), T_Q(u_i)\right)}\right]\right)$$

$$S_{2F}(P,Q) = \frac{1}{n}\left(\sum_{1}^{N}\left[\frac{min\left(I_P(u_i), I_Q(u_i)\right)}{max\left(I_P(u_i), I_Q(u_i)\right)}\right]\right)$$

$$S_{2F}(P,Q) = \frac{1}{n}\left(\sum_{1}^{N}\left[\frac{min\left(F_P(u_i), F_Q(u_i)\right)}{max\left(F_P(u_i), F_Q(u_i)\right)}\right]\right)$$

and $S_1 = \left(S_{2T}(P,Q), S_{2I}(P,Q), S_{2F}(P,Q)\right)$. (2.6)

3. Similarity Measure of Neutrosophic Sets

In this section, we mainly introduced some new similarity measures for the single valued neutrosophic sets based on the exponential function. Let U be the universe of discourse.

Definition 5 Consider $P = \{(T_P(u_i), I_P(u_i), F_P(u_i))|u_i\epsilon U\}$ and $Q = \{(T_Q(u_i), I_Q(u_i), F_Q(u_i))|u_i\epsilon U, i = 1,2,\dots,n\}$ be two valued neutrosophic sets, then the similarity measure $SM_1(P,Q)$ between P and Q is defined as:

$$SM_1(P,Q) = \frac{1}{n}\sum_{i=1}^{n}\left(SM_i^T(u_i) \times SM_i^I(u_i) \times SM_i^F(u_i)\right); \quad (3.1)$$

where $SM_i^T(u_i) = e^{-|T_P(u_i) - T_Q(u_i)|}$; $SM_i^I(u_i) = e^{-|I_P(u_i) - I_Q(u_i)|}$ & $SM_i^F(u_i) = e^{-|F_P(u_i) - F_Q(u_i)|}$.

Definition 6 Consider $P = \{(T_P(u_i), I_P(u_i), F_P(u_i))|u_i\epsilon U\}$ and $Q = \{(T_Q(u_i), I_Q(u_i), F_Q(u_i))|u_i\epsilon U, i = 1,2,\dots,n\}$ be two valued neutrosophic sets, then the weighted similarity measure $SM_1^w(P,Q)$ between P and Q is defined as:

$$SM_1^w(P,Q) = \sum_{i=1}^{n} w_i \times \left(SM_i^T(u_i) \times SM_i^I(u_i) \times SM_i^F(u_i)\right); \quad (3.2)$$

where $SM_i^T(u_i) = e^{-|T_P(u_i) - T_Q(u_i)|}$; $SM_i^I(u_i) = e^{-|I_P(u_i) - I_Q(u_i)|}$ & $SM_i^F(u_i) = e^{-|F_P(u_i) - F_Q(u_i)|}$.

Definition 7 Suppose $P = \{(T_P(u_i), I_P(u_i), F_P(u_i))|u_i\epsilon U\}$ and

$Q = \{(T_Q(u_i), I_Q(u_i), F_Q(u_i))|u_i\epsilon U, i = 1,2, \dots, n\}$ be two valued neutrosophic sets, then the similarity measure $SM_2(P,Q)$ between P and Q is defined as:

$$SM_2(P,Q) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{SM_i^T(u_i)+SM_i^I(u_i)+SM_i^F(u_i)}{3}\right); \tag{3.3}$$

where $SM_i^T(u_i) = e^{-|T_P(u_i)-T_Q(u_i)|}$; $SM_i^I(u_i) = e^{-|I_P(u_i)-I_Q(u_i)|}$ & $SM_i^F(u_i) = e^{-|F_P(u_i)-F_Q(u_i)|}$.

Definition 8 Consider $P = \{(T_P(u_i), I_P(u_i), F_P(u_i))|u_i\epsilon U\}$ and $Q = \{(T_Q(u_i), I_Q(u_i), F_Q(u_i))|u_i\epsilon U, i = 1,2, \dots, n\}$ be two valued neutrosophic sets, then the weighted similarity measure $SM_2^w(P,Q)$ between P and Q is defined as:

$$SM_2^w(P,Q) = \sum_{i=1}^{n} w_i \times \left(\frac{SM_i^T(u_i)+SM_i^I(u_i)+SM_i^F(u_i)}{3}\right); \tag{3.4}$$

where $SM_i^T(u_i) = e^{-|T_P(u_i)-T_Q(u_i)|}$; $SM_i^I(u_i) = e^{-|I_P(u_i)-I_Q(u_i)|}$ & $SM_i^F(u_i) = e^{-|F_P(u_i)-F_Q(u_i)|}$

Theorem 1 The measure proposed in Definition 5 is a valid similarity measure.

Proof: For this, we need to show that the similarity measure $SM_1(P,Q)$ between two neutrosophic sets P and Q holds the conditions as defined in Definition 4.

(i) We know that $T_P(u_i), T_Q(u_i) \leq 1$, which implies $|T_P(u_i) - T_Q(u_i)| \leq 1$. This can also be written as $-1 \leq |T_P(u_i) - T_Q(u_i)| \leq 0$.

Hence,

$$0 \leq e^{-|T_P(u_i)-T_Q(u_i)|} \leq 1 \Rightarrow 0 \leq SM_i^T(u_i) \leq 1.$$

Also $0 \leq SM_i^I(u_i), SM_i^F(u_i) \leq 1$. Therefore, from equation (3.1) we conclude that $0 \leq SM_1(P,Q) \leq 1$.

(ii) We know that $SM_i^T(u_i) = 1$, $SM_i^I(u_i) = 1$ and $SM_i^F(u_i) = 1$ if and only if $P = Q$, so we have $SM_1(P,Q) = 1 \Leftrightarrow P = Q$.

(iii) As $SM_i^T(u_i), SM_i^I(u_i), SM_i^F(u_i)$ are symmetric for neutrosophic sets. Hence, we observe that $SM_1(P,Q) = SM_1(Q,P)$.

(iv) If $P \subseteq Q \subseteq O$, then for $u_i \in U$ we have

$$0 \leq T_P(u_i) \leq T_Q(u_i) \leq T_O(u_i) \leq 1;$$
$$0 \geq I_P(u_i) \geq I_Q(u_i) \geq I_O(u_i) \geq 1;$$

and

$$0 \leq F_P(u_i) \leq F_Q(u_i) \leq F_O(u_i) \leq 1.$$

It means that

$$-|T_P(u_i) - T_Q(u_i)| \leq \min\{|T_P(u_i) - T_Q(u_i)|, |T_Q(u_i) - T_O(u_i)|\};$$
$$-|I_P(u_i) - I_Q(u_i)| \leq \min\{|I_P(u_i) - I_Q(u_i)|, |I_Q(u_i) - I_O(u_i)|\};$$

and

$$-|F_P(u_i) - F_Q(u_i)| \leq \min\{|F_P(u_i) - F_Q(u_i)|, |F_Q(u_i) - F_O(u_i)|\};$$

This implies that

$$SM_i^T(P,Q) \leq \min\{SM_i^T(P,Q), SM_i^T(Q,O)\};$$
$$SM_i^I(P,Q) \leq \min\{SM_i^I(P,Q), SM_i^I(Q,O)\};$$

and

$$SM_i^F(P,Q) \leq \min\{SM_i^F(P,Q), SM_i^F(Q,O)\}.$$

Thus, based on this, equation (3.1) becomes $SM_1(P,Q) \leq SM_1(P,Q)$ and $SM_1(P,Q) \leq SM_1(Q,O)$. Hence, the proposed measure in the Definition 5 is the valid similarity measure over two neutrosophic sets.

Theorem 2 The measure proposed in the Definition 6 is a valid similarity measure.

Proof: For this, we need to show the similarity measure $SM_1(P,Q)$ between two neutrosophic sets P and Q holds the conditions defined in Definition 4.

(i) We know that $T_P(u_i), T_Q(u_i) \leq 1$, which implies $|T_P(u_i) - T_Q(u_i)| \leq 1$. This can also be written as

$$-1 \leq |T_P(u_i) - T_Q(u_i)| \leq 0.$$

Hence, $0 \leq e^{-|T_P(u_i) - T_Q(u_i)|} \leq 1 \Rightarrow 0 \leq SM_i^T(u_i) \leq 1$.

Also, $0 \leq SM_i^I(u_i), SM_i^F(u_i) \leq 1$.

Therefore, from equation (3.1) we conclude that

$$0 \leq SM_1^w(P,Q) \leq \sum_{i=1}^{n} w_i = 1.$$

(ii) We know that $SM_i^T(u_i) = 1$, $SM_i^I(u_i) = 1$ and $SM_i^F(u_i) = 1$ if only if $P = Q$ because, $\sum_{i=1}^{n} w_i = 1$, so we have , $SM_1^w(P,Q) = 1 \Leftrightarrow P = Q$.

(iii) As $SM_i^T(u_i)$, $SM_i^I(u_i)$, $SM_i^F(u_i)$ are symmetric for neutrosophic sets. Hence, we observe that $SM_1^w(P,Q) = SM_1(Q,P)$.

(iv) For $P \subseteq Q \subseteq O$ and $u_i \in U$, we have

$$SM_i^T(P,Q) \leq \min\{SM_i^T(P,Q), SM_i^T(Q,O)\};$$

$$SM_i^I(P,Q) \leq \min\{SM_i^I(P,Q), SM_i^I(Q,O)\};$$

and

$$SM_i^F(P,Q) \leq \min\{SM_i^F(P,Q), SM_i^F(Q,O)\}.$$

Thus, based on this, equation (3.2) becomes $SM_1^w(P,Q) \leq SM_1^w(P,Q)$ and $SM_1^w(P,Q) \leq SM_1^w(Q,O)$.

Hence, the proposed measure in the Definition 6 is the valid similarity measure over two neutrosophic sets.

3.1 Comparison with Existing Similarity Measures

In order to show the effectiveness, performance and advantages of the proposed similarity measures, we present the following comparative analysis with existing measures presented in Equation (2.1), Equation (2.2), Equation (2.3) and Equation (2.4).

Thus, to carry out the comparison of the proposed similarity measures with the existing ones in the literature, we consider five different cases consisting of two neutrosophic sets as follows:

$$\text{Case 1: A} = \{0.2,0.3,0.4\} \ \& \ \text{B} = \{0.2, 0.3, 0.4\}$$

$$\text{Case 2: A} = \{0.3,0.2,0.4\} \ \& \ \text{B} = \{0.4, 0.2, 0.3\}$$

$$\text{Case 3: A} = \{1,0.0,0.0\} \ \& \ \text{B} = \{0.0, 1, 1\}$$

$$\text{Case 4: A} = \{1,0.0,0.0\} \ \& \ \text{B} = \{0.0, 0.0, 0.0\}$$

$$\text{Case 5: A} = \{0.4,0.2,0.6\} \ \& \ \text{B} = \{0.2, 0.1, 0.3\}$$

Based on the computational analysis, the values obtained by the proposed similarity measures and existing similarity measures for each case have been tabulated in the Table 1.

Table 1: Comparison of Proposed Similarity Measure with Existing One's

	Case 1	**Case 2**	**Case 3**	**Case 4**	**Case 5**
SM_1	1	0.8187	0.0497	0.3678	0.5488
SM_2	1	0.978	0.3678	0.7892	0.8214
$S_j[28]$	1	0.93	0.0	0.0	0.666
$S_D[28]$	1	0.965	0.0	0.0	0.8
$S_C[28]$	1	0.965	0.0	Null	1
$S_T[28]$	1	-2.10	0.954	0.984	0.259

In view of the computed values obtained by the different measures, we can conclude that the proposed similarity measures are quite effective and give distinguished result whereas the existing ones are not able to perform good in some cases (indicated by the bold values).

Remark: "Null" represents the case when the degree of similarity can not be computed due to the problem "division by zero".

4. Applications of Neutrosophic Similarity Measures

4.1 Classification Problem

Consider a standard classification problem where we have m different classes (say) $C_1, C_2, C_3, \dots, C_m$ of known patterns over the universe of discourse $U = \{u_1, u_2, u_3, \dots, u_n\}$. Suppose we choose one sample (say) $P_1, P_2, P_3, \dots, P_m$ from each class and have an unknown sample Q where the information in each known and unknown pattern is featured under the neutrosophic environment. Thus, our main objective is to classify the unknown sample into one of the known classes.

In order to solve this classification problem, we calculate the similarity measure of unknown sample Q with each known pattern $P_i(i = 1, 2, 3, \dots, m)$ and then allocate the unknown sample to one of the classes which has highest similarity index among all.

Example 1: Let us consider three existing patterns P_1, P_2 and P_3 being described by the neutrosophic sets in $U = \{u_1,\ u_2, u_3\}$ as following:

$$P_1 = \{(u_1, 0.5, 0.4, 0.2), (u_2, 0.4, 0.3, 0.4), (u_3, 0.4, 0.5, 0.1)\};$$

$$P_2 = \{(u_1, 0.6, 0.5, 0.1), (u_2, 0.5, 0.1, 0.3), (u_3, 0.5, 0.5, 0.1)\};$$

$$P_3 = \{(u_1, 0.4, 0.4, 0.2), (u_2, 0.4, 0.5, 0.2), (u_3, 0.3, 0.3, 0.4)\};$$

Let us take an unknown pattern Q given by

$$Q = \{(u_1, 0.4, 0.4, 0.2), (u_2, 0.5, 0.6, 0.1), (u_3, 0.3, 0.4, 0.4)\}.$$

Now, the main task to be accomplished in the problem is to find the class to which Q belongs.

We present the computational procedure of solving the classification problem under consideration with the help of following Figure 2.

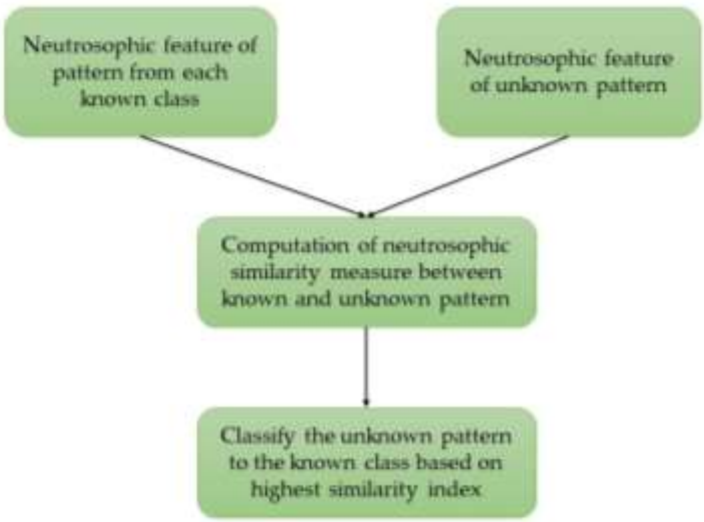

Figure 2: Computational Procedure for Classification Problem

With the help of proposed similarity measures given by equations (3.1), (3.2), (3.3) & (3.4), and choosing the arbitrary weight vector $\boldsymbol{w} = (\mathbf{0.3}, \mathbf{0.4}, \mathbf{0.3})$ (may be selected on the decision maker's choice) of the elements of U, we compute the desired values and tabulate them in Table 2.

Table 2: Computed Values of Similarity Measures

	$(\boldsymbol{P_1}, \mathbf{Q})$	$(\boldsymbol{P_2}, \mathbf{Q})$	$(\boldsymbol{P_3}, \mathbf{Q})$

SM_1	0.6725	0.5611	0.5322
SM_1^w	0.6659	0.5656	0.5530
SM_2	0.880	0.8226	0.804
SM_2^w	0.876	0.824	0.814

Based on the obtained values in Table 2, we conclude that the unknown pattern Q belongs to the class $\boldsymbol{P_1}$. The results obtained by utilizing the proposed similarity measures are certainly found to be consistent with the results obtained in [2]. The values obtained are also more prominent and decisive in nature.

4.2 Evaluation Process in Decision Making

In view of the general format of a decision-making problem, we consider a set of available alternatives (say) $\{\boldsymbol{Z_1}, \boldsymbol{Z_2}, \dots, \boldsymbol{Z_m}\}$ and the set of criteria (say) $\boldsymbol{O_1}, \boldsymbol{O_2}, \dots, \boldsymbol{O_n}$. The main goal of the problem is to select the optimal and the best alternatives out of the m available alternatives with respect to n criteria.

The procedure for ranking of the alternatives is based on transforming the neutrosophic decision matrix and computing the similarity index between the alternatives and the ideal solution which has been clearly represented with the help of the following block diagram given in Figure 3:

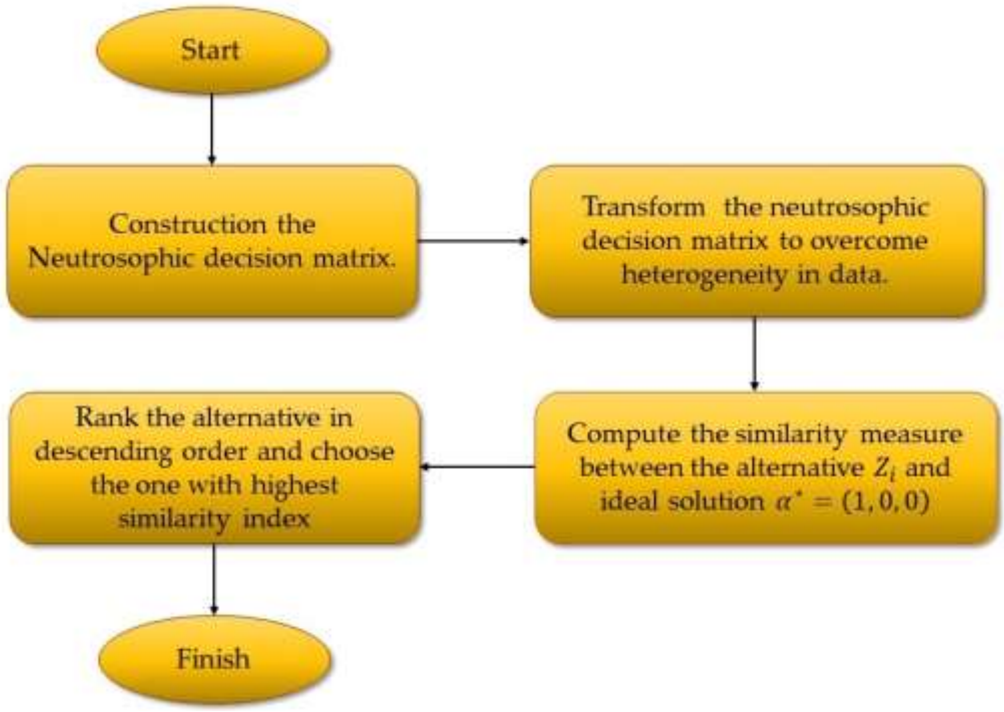

Figure 3: Ranking Procedure for Decision Making with Similarity Measures

Example 2: Consider there is a financial private limited firm whose objective is to invest a significant amount of money in the best possible sector. Suppose there are four possible investment sectors selected on the basis of an initial survey, say,

- $\boldsymbol{Z_1}$: Automobile Sector,
- $\boldsymbol{Z_2}$: Food & Beverages Service Sector,
- $\boldsymbol{Z_3}$: Information Technology Sector,
- $\boldsymbol{Z_4}$: Ammunition Production Sector.

The investment company must take a decision according to the following three important criteria:

- $\boldsymbol{O_1}$: Risk Factor,
- $\boldsymbol{O_2}$: Growth Prospects,
- $\boldsymbol{O_3}$: Ecological Impact.

Suppose that the management and the decision-makers assign suitable weights to each criteria based on their experience and risk bearing capability given by $w = (0.35,0.25,0.4)$. The necessary information has been taken from the experts/decision makers for the sake of evaluation of the alternatives $Z_i's$ with respect to each criterion $O_j's$.
The opinion values of each alternative with respect to each criteria have been expressed as a neutrosophic information, and the following neutrosophic decision matrix has been provided:

$$\mathcal{R} = \begin{array}{c} \\ \mathcal{Z}_1 \\ \mathcal{Z}_2 \\ \mathcal{Z}_3 \\ \mathcal{Z}_4 \end{array} \begin{array}{c} \begin{array}{ccc} O_1 & O_2 & O_3 \end{array} \\ \begin{bmatrix} (0.4, 0.2, 0.3) & (0.4, 0.2, 0.3) & (0.2, 0.2, 0.5) \\ (0.6, 0.1, 0.2) & (0.6, 0.1, 0.2) & (0.5, 0.2, 0.2) \\ (0.3, 0.2, 0.3) & (0.5, 0.2, 0.3) & (0.5, 0.3, 0.2) \\ (0.7, 0.0, 0.1) & (0.6, 0.1, 0.2) & (0.4, 0.3, 0.2) \end{bmatrix} \end{array}$$

The ideal solution in such decision-making problems can be as $\boldsymbol{\alpha}^* = (\mathbf{1}, \mathbf{0}, \mathbf{0})$. However, it may be noted that the ideal solution generally does not exist in practice but a closer value is accepted. Our decision can be obtained by calculating the values proposed similarity measures between each alternative $\boldsymbol{Z_i}(\boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4})$ and the ideal solution $\boldsymbol{\alpha}^*$. In view of the procedure presented in Figure 3, these values have been computed and tabulated in the Table 3.

Table 3: Computed values of Similarity measure

	$\boldsymbol{SM_1}$	$\boldsymbol{SM_1^w}$	$\boldsymbol{SM_2}$	$\boldsymbol{SM_2^w}$
$(\boldsymbol{Z_1}, \boldsymbol{\alpha}^*)$	0.2962	0.2889	0.6768	0.6716
$(\boldsymbol{Z_2}, \boldsymbol{\alpha}^*)$	0.4665	0.4605	0.7813	0.7779
$(\boldsymbol{Z_3}, \boldsymbol{\alpha}^*)$	0.3456	0.3445	0.7098	0.7092
$(\boldsymbol{Z_4}, \boldsymbol{\alpha}^*)$	0.6703	0.4919	0.7942	0.7892

On the basis of the computed values, the ranking order of the four alternatives in the above problem is

$$Z_4 > Z_2 > Z_3 > Z_1$$

Thus, we have that the alternative Z_4 is the best choice among all the alternatives. The results obtained by utilizing the proposed similarity measures are consistent with the results obtained by Ye [3] and Wang et al. [1].

5. Conclusions & Scope for Future Work

We have successfully introduced some new measures of similarity for the neutrosophic sets in terms of the exponential functions of the truth membership, indeterminacy-membership and falsity-membership. The efficiency of the proposed measure has been validated by presenting few counter-intuitive cases which show that the existing measures fail under some certain cases, while the proposed measures classify them more accurately and precisely. Furthermore, to illustrate the applicability of the proposed similarity measures, an example of classification problem and an example of decision-making problem under neutrosophic environment have been successfully solved. Finally, we conclude that the proposed types of exponential similarity measures are better than the existing measures. The proposed measures produce a reasonable and distinguishable results which is the main outcome and advantage in contrast with other existing methods. Also, it may clearly be observed that the proposed measures are very simple and have the minimum computational burden as compared with other existing methods. The psoposed exponenital similarity measure for the the neutrosophic sets can be extended for single and multi-valued neutrosophic hypersoft set also along with the relvant application which will certainly give an added advantage in the literature. The proposed strategy utilizing the exponential similarity measure can further be applied in various other decision-making problems.

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Neutrosophic Statistical Analysis of Income of YouTube Channels

Ishmal Shahzadi[1], **Muhammad Aslam**[2*] and **Hussain Aslam**[3]

[1]College of Statistical and Actuarial Sciences, University of the Punjab, Lahore, Pakistan.
Email: ishahzadiaslam@gmail.com

[2]Department of Statistics, Faculty of Science, King Abdulaziz University, Jeddah 21551, Saudi Arabia; email: aslam_ravian@hotmail.com

[3]Pakistan International School, Jeddah 21551, Saudi Arabia; email: hussainaslam059@gmail.com

*corresponding author email: aslam_ravian@hotmail.com

Abstract: The YouTube website is famous among all ages of people due to the verity of purposes. YouTube is also a good platform for earning; therefore, numbers of channels are operating through this website. The income of each channel is directly proportional to the number of views, likes, and subscribers. In this paper, three famous channels including Ducky Bahi TV, Hasi TV and Haqeeqat TV are chosen and their income is recorded. The analysis of these channels using neutrosophic statistics is given. From the analysis, it is concluded that on the average Haqeeqat TV is better in earning. On the other hand, Hasi TV is more consistent in earning as compared to other channels.

Keywords: Social media; YouTube; indeterminacy; income; statistics

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1. Introduction

There is no life without the internet nowadays. The YouTube channel is very famous among children, young men, and even old men. This channel is a big source of knowledge and income. The different channels on YouTube provide information on political, games, and current affairs. This YouTube gives the opportunity to channels to earn money is based on the number of views, subscribers, and likes. The channel gets more income as the number of subscribers is increased. In nutshell, the income of the YouTube channel is directly proposal to the number of subscribers. This website is the 3rd position among the web sites of the world. Due to the attractive packages of YouTube, the people are launching their channels to entertain the audience. Due to the popularity of YouTube channels, several researchers studied different aspects of YouTube. Reference [1] presented a study on the influence of YouTube on the big cities. Reference [2] worked on the effect of YouTube on academic performance. Reference [3] studied its contents published in different journals. Reference [4] discussed the limitations of the channel. References [1] and [5] presented a statistical analysis for views and uploads of this website. [4] presented the statistical analysis of educational videos. Reference [6] presented the analysis of childhood behavior. Reference [7] studied the effect of advertisements on the purchases. The existing literature assumes that the observations in the data obtained from YouTube should be determined, exact, and precise. In practice, the social media data is not always determined but in intervals. For example, the estimated income of each YouTube channel is in interval rather than the exact number. In such cases, the analysis is done using the fuzzy-based approach. References [8] and [9] presented the analysis techniques for the fuzzy data. References [10], [11] and [12] developed the statistical test for fuzzy data. More details about the fuzzy-based analysis can be seen in [13], [14] and [15].

[16] introduced the neutrosophic logic to deals with the measure of intermediacy. The neutrosophic logic is more efficient than the fuzzy logic, see [17]. The various applications of this logic can be viewed in [18], [19], [20], [21] and [22]. To deal with the neutrosophic data or data in the interval [23] introduced the neutrosophic statistics (NS). The NS is more efficient than classical statistics (CS). References [24] and [25] presented the methods to analyze the neutrosophic data. [26] and [27] proposed a test using NS.

According to [23], [28] and [29] (Smarandache 2014, 2016-2019), "while the Classical Statistics deals with *determinate data* and *determinate inference methods* only, Neutrosophic Statistics deals with *indeterminate data* and *indeterminate inference methods*, i.e. data that has any kind of indeterminacy (unclear, vague, partially unknown, contradictory, incomplete, etc.), and inference methods that degrees of indeterminacy as well (for example, instead of crisp arguments and values for the probability distributions, algorithms, functions etc. one may have inexact or ambiguous arguments and values). Neutrosophic Statistics was founded by Smarandache in 1998 and developed in 2014. The Neutrosophic Statistics is also a generalization of Interval Statistics, because of, among others, while Interval Statistics is based on Interval Analysis, Neutrosophic Statistics is based on Set Analysis (meaning all kind of sets, not only intervals). Neutrosophic Statistics is more elastic than Classical Statistics. If all data and inference methods are determinate, then Neutrosophic Statistics coincides with Classical Statistics. But, since in our world we have more indeterminate data than determinate data, therefore more neutrosophic statistical procedures are needed than classical ones."

The NS provides information about the measure of indeterminacy when the data is obtained under uncertainty. By exploring the literature, the NS was not applied to analyze the income of YouTube channels. In this paper, we will select three famous YouTube channels including Ducky Bahi TV, Hasi TV, and Haqeeqat TV. We will record the income of these channels and present the analysis using the idea of NS. From the analysis, it can be expected that the proposed NS analysis will be more informative than the analysis based CS.

2. Method

Suppose that $X_N = X_L + X_U I_N; I_N \epsilon [I_L, I_U]$ be a neutrosophic random variable which represents the income of YouTube channels, where X_L is the lower-income and $X_U I_N$ is the upper-income level and $I_N \epsilon [I_L, I_U]$ be the indeterminacy interval. By following [24], the neutrosophic average $\bar{X}_N \epsilon [\bar{X}_L, \bar{X}_U]$ income can be calculated as

$$\bar{X}_N = \bar{X}_L + \bar{X}_U I_N;\ I_N \epsilon [I_L, I_U] \quad (1)$$

where $\bar{X}_L = \frac{1}{n_N}\sum_{i=1}^{n_N} X_{iL}$, $\bar{X}_U = \frac{1}{n_N}\sum_{i=1}^{n_N} X_{iU}$ and $n_N \epsilon [n_L, n_U]$ be a neutrosophic random sample. The neutrosophic sum of square (NSS) can be computed as follows

$$\sum_{i=1}^{n_N}(X_i - \bar{X}_{iN})^2 = \sum_{i=1}^{n_N}\begin{bmatrix} \min\begin{pmatrix} (a_i + b_i I_L)(\bar{a} + \bar{b} I_L), (a_i + b_i I_L)(\bar{a} + \bar{b} I_U) \\ (a_i + b_i I_U)(\bar{a} + \bar{b} I_L), (a_i + b_i I_U)(\bar{a} + \bar{b} I_U) \end{pmatrix} \\ \max\begin{pmatrix} (a_i + b_i I_L)(\bar{a} + \bar{b} I_L), (a_i + b_i I_L)(\bar{a} + \bar{b} I_U) \\ (a_i + b_i I_U)(\bar{a} + \bar{b} I_L), (a_i + b_i I_U)(\bar{a} + \bar{b} I_U) \end{pmatrix} \end{bmatrix}, I\epsilon[I_L, I_U] \quad (2)$$

where $a_i = X_L$ and $b_i = X_U$. The neutrosophic sample variance can be computed by

$$S_N^2 = \frac{\sum_{i=1}^{n_N}(X_i - \bar{X}_{iN})^2}{n_N}; S_N^2 \epsilon [S_L^2, S_U^2] \quad (3)$$

The neutrosophic coefficient of variation (NCV) measures the consistency of the YouTube channels. The smaller value of NCV means, the performance of the YouTube channel is more consistent than the other channels. The NCV can be computed by

$$CV_N = \frac{\sqrt{S_N^2}}{\bar{X}_N} \times 100;\ CV_N \epsilon [CV_L, CV_U] \quad (4)$$

3. Data Collection

As we know, thousands of channels are operating through YouTube websites that can be selected to study various aspects. Here, we will select only the three famous channels including Ducky Bahi TV, Hasi TV, and Haqeeqat TV. The available latest data about the estimated income of two weeks is selected to discuss the analysis under Neutrosophy. The same approach can be applied for any number of days and channels. It can be noted that the estimated income of these channels has the lower limit and upper limit. The income data is expressed in intervals rather than the exact income value. For example, on the date 29-6-2020, the estimated income for Haqeeqat TV is 25$ to 406$. The income data of these channels are shown in Table 1.

Table 1: The income of three channels

Date	Ducky Bahi TV	Hasi TV	Haqeeqat TV
2020-06-16	[102,1600]	[28,452]]	[348,5600]
2020-06-17	[86,1400]	[32,505]	[407,6500]
2020-06-18	[83,1300]	[30,477]	[414,6600]
2020-06-19	[75,1200]	[25,392]	[419,6700]
2020-06-20	[65,1000]	[35,567]	[395,6300]`
2020-06-21	[63,1000]	[36,571]	[367,5900]
2020-06-22	[66,1100]	[43,692]	[362,5800]
2020-06-23	[61,972]	[46,730]	[357,5700]
2020-06-24	[57,917]	[48,774	[337,5400]
2020-06-25	[56,893]	[41,661]	[347,5600]
2020-06-26	[58,928]	[40,643]	[386,6200]
2020-06-27	[57,905]	[43,685,]	[380,6100]
2020-06-28	[209,3300]	[44,705]	[371,5900]

2020-06-29	[140,2200]	[35,566]	[25,406]

We note that the estimated income of the channel is expressed in the interval. For the interval data, the application of classical statistics is not suitable and informative. The use of classical statistics does not give us information about the measure of indeterminacy associated with these income intervals. Therefore, according to the nature of the income data of these channels, the analysis under neutrosophic statistics can be effective, suitable, and informative.

4. Neutrosophic Statistical Analysis

In this section, the application of neutrosophic statistics is given using the income data of three channels is given in Table 1. To see that which channel is better on the average in income, the values of neutrosophic averages $\bar{X}_N \epsilon [\bar{X}_L, \bar{X}_U]$ are computed and presented in Table 2. To study the variation in income, the values of neutrosophic standard deviation $S_N \epsilon [S_L, S_U]$ are also presented in Table 2. To see which channel is more consistent, the values of $CV_N \epsilon [CV_L, CV_U]$ are computed and given in Table 2.

Table 2: Neutrosophic Descriptive Statistics

TV Channels	$\bar{X}_N$	S_N	CV_N
Ducky Bahi TV	[84.14, 1336.78]	[41.18,686.77]	[48.94,51.38]
Hasi Tv	[37.57,601.42]	[6.82,117.60]	[18.15,19.55]
Haqeeqat TV	[351.07,5621.85]	[93.74,27022.69]	[26.70,480.67]

From Table 2, it can be observed that Haqeeqat TV has higher values of income as compared to Hasi TV and Ducky Bahi TV. It means, Haqeeqat TV is better on average than the other channels. On the other hand, the value of CV_N for Hasi TV is lower than the other channels. It means, Hasi TV performance in income-earning is consistent than the other channels. From this study, it can be concluded that although Haqeeqat TV is better on the average Hasi TV is more consistent than the other channels.

5. Comparative Study

To show that the neutrosophic statistical the analysis is more informative than classical statistics, the neutrosophic form and associated measure of indeterminacy for $\bar{X}_N \epsilon [\bar{X}_L, \bar{X}_U]$, $S_N \epsilon [S_L, S_U]$ and $CV_N \epsilon [CV_L, CV_U]$ are given in Table 3. For example, the neutrosophic form of $CV_N \epsilon [CV_L, CV_U]$ for Hasi TV is:

$CV_N = 18.15 + 19.55 I_N; I_N \epsilon [0, 0.07]$. The first value of this neutrosophic form presents the analysis from the classical statistics. The neutrosophic result reduces to classical statistics results when $I_N = 0$. The second part of the neutrosophic form presents the upper value of neutrosophic interval. From this neutrosophic form, it can be seen that the values of CV_N is from 18.15% to 19.55% with the measure of intermediacy 0.07. From Table 3, it can be seen that measure of indeterminacy is smaller for Hasi TV. It means that the smaller the value CV_N has smaller the value of the measure of indeterminacy.

Table 3: Neutrosophic forms with measure of indeterminacy

TV Channels	$\bar{X}_N$	S_N	CV_N
Ducky Bahi TV	$84.14 + 1336.78I_N; I_N\epsilon[0,0.94]$	$41.18 + 686.77I_N; I_N\epsilon[0,0.94]$	$48.94 + 51.38I_N; I_N\epsilon[0,0.08]$
Hasi TV	$37.57 + 601.42I_N; I_N\epsilon[0,0.94]$	$6.82 + 117.60I_N; I_N\epsilon[0,0.94]$	$18.15 + 19.55I_N; I_N\epsilon[0,0.07]$
Haqeeqat TV	$351.07 + 5621.85I_N; I_N\epsilon[0,0.94]$	$93.74 + 27022.69I_N; I_N\epsilon[0,0.99]$	$26.7 + 480.67I_N; I_N\epsilon[0,0.94]$

6. Concluding Remarks

The YouTube website was famous among all ages of people due to a verity of purposes. YouTube is also a good platform for earning; therefore, numbers of channels were operating through this website. The income of each channel was directly proportional to the number of views, likes, and subscribers. In this paper, three famous channels including Ducky Bahi TV, Hasi TV, and Haqeeqat TV were chosen and their income was recorded. The analysis of these channels using neutrosophic statistics was given. From the analysis, it was concluded that on the average Haqeeqat TV is better in earning. On the other hand, Hasi TV was more consistent in earning as compared to other channels. From the neutrosophic statistical analysis, it can be concluded that a channel having the smaller value of CV has a smaller value of the measure of indeterminacy. Using the same neutrosophic analysis, various aspects of channels can be studied as future research.

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Semi Homomorphisms and Algebraic Relations Between Strong Refined Neutrosophic Modules and Strong Neutrosophic Modules

[1]Mohammad Abobala

[1]Faculty of Science,, Department of Mathematics Tishreen University, Lattakia Syria

[1]e-mail: mohammadabobala777@gmail.com

Abstract: The objective of this paper is to define and study the concept of semi homomorphism between two modules defined over different rings. Semi homomorphisms can help in the study of algebraic relations between two modules defined over two different rings. An application of this functions is presented in the case of refined strong neutrosophic modules and neutrosophic strong modules, where strong neutrosophic modules are shown to be semi homomorphic images of their corresponding strong refined neutrosophic modules. Also, this work presents a discussion of the algebraic structure of some semi homomorphisms and isomorphisms.

Keywords: Neutrosophic module, $f-$ semi homomorphism, $f-$ semi isomorphism, neutrosophic ring, refined neutrosophic ring, classical homomorphism..

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1. Introduction

Neutrosophy as a new kind of logic founded by Smarandache plays an important role in many fields of knowledge as well as algebra. We find many applications of neutrosophic sets in decision making [20,23], medical studies [14], and computer science [21]. Also, many algebraic neutrosophic structures were defined and handled such as neutrosophic rings and neutrosophic modules/spaces [1,6,7,8,18].

Recently, F. Smarandache came with an interesting idea suggests that the indeterminacy I can be split into two different degrees of indeterminacy I_1, I_2. Agboola et. al used this idea to introduce the concept of refined neutrosophic ring [2,4]. Substructures of these rings were studied widely in [1,2,3,5,6,7,10,11] such as neutrosophic isomorphisms, AH-ideals, AHS-homomorphisms, idempotents, and semi idempotents.

Neutrosophic modules were firstly defined over neutrosophic rings in [8] and studied widely in [4], and then they were generalized into refined neutrosophic modules over refined neutrosophic rings in [9]. Recently, they were generalized into n-refined neutrosophic modules [16] over n-refined neutrosophic rings [19].

It is well known that the relationships between two modules can be represented with module homomorphisms if these modules were defined over the same ring. In this work we introduce the concept of semi homomorphism between two modules to help us in the study of the relationships between two different modules defined over two different rings. Our goal is to study the relationships between strong refined neutrosophic modules and strong neutrosophic modules by using the notion of semi homomorphisms.

All rings through this paper are considered commutative with unity 1. Also, all neutrosophic modules and refined neutrosophic modules are considered strong.

2. Preliminaries

Definition 2.1: [6]

Let (R,+,×) be a ring, $R(I) = \{a + bI \,;\, a, b \in R\}$ is called the neutrosophic ring where I is a neutrosophic element with condition $I^2 = I$.

Remark 2.2 : [5]

The element I can be split into two indeterminacies I_1, I_2 with conditions :

$$\mathrm{I}_1{}^2 = \mathrm{I}_1 \,, I_2{}^2 = I_2 \,, I_1 I_2 = I_2 I_1 = I_1 \,.$$

Definition 2.3 : [5]

If X is a set, then $X(I_1, I_2) = \{(a, bI_1, cI_2); a, b, c \in X\}$ is called the refined neutrosophic set generated by X , I_1, I_2.

Definition 2.4 : [6]

Let (R,+,×) be a ring, $(R(I_1, I_2), +, \times)$ is called a refined neutrosophic ring generated by R , I_1, I_2.

Theorem 2.5: [6]

Let $(R(I_1, I_2), +, \times)$ be a refined neutrosophic ring then it is a ring .

Definition 2.6:[8]

Let (M,+,.) be a module over the ring R, then (M(I),+,.) is called a weak neutrosophic module over the ring R, and it is called a strong neutrosophic module if it is a module over the neutrosophic ring R(I). Elements of M(I) have the form $x + yI; x, y \in M$, i.e M(I) can be written as $M(I) = M + MI$.

Definition 2.7: [8]

Let M(I) be a strong neutrosophic module over the neutrosophic ring R(I) and W(I) be a non empty subset of M(I), then W(I) is called a strong neutrosophic submodule if W(I) itself is a strong neutrosophic module.

Definition 2.8:[8]

Let U(I) and W(I) be two strong neutrosophic submodules of M(I) and let $f: U(I) \to W(I)$, we say that f is a neutrosophic vector space homomorphism if

(a) $f(I) = I$.

(b) f is a module homomorphism.

We define the kernel of f by Ker f = { x∈ $M(I)$; $f(x) = 0$ }.

Definition 2.9: [9]

Let $(X(I_1, I_2), +, \cdot)$ be any refined neutrosophic algebraic structure where + and $\cdot$ are ordinary addition and multiplication respectively. I_1 *and* I_2 are the split components of the indeterminacy factor I that is $I = \alpha I_1 + \beta I_2$ with $\alpha, \beta \in R$ *or* C. Also, I_1 *and* I_2 are taken to have the properties $I_1^2 = I_1, I_2^2 = I_2$ and $I_1 I_2 = I_2 I_1 = I_1$.

For any two elements, we define

1) $x + y = (a, bI_1, cI_2) + (d, eI_1, fI_2) = (a + d, (b + e)I_1, (c + f)I_2)$

2) $x \cdot y = (a, bI_1, cI_2) \cdot (d, eI_1, fI_2) = \begin{pmatrix} ad, (ae + bd + be + bf + ce)I_1, \\ (af + cd + cf)I_2 \end{pmatrix}$

Definition 2.10: [9]

Let $(M,+,.)$ be any R-module over a refined neutrosophic ring $R(I_1,I_2)$, The triple $(M(I_1,I_2),+,\cdot)$ is called a strong refined neutrosophic R-module over a refined neutrosophic ring $R(I_1,I_2)$, generated by M, I_1 and I_2.

Theorem 2.11: [3]

Let (R ,+,×) be a ring and R(I), R(I_1,I_2) the related neutrosophic ring and refined neutrosophic ring respectively, we have:

(a) There is a ring homomorphism f: R$(I_1,I_2) \to R(I)$.

(b) The additive group (Ker f , +) is isomorphic to the additive group (R,+).

3. Main discussion

Definition 3.1:

Let M be a module over a ring R, N be a module over a ring T, $\varphi: M \to N$ be a well defined map, we say that φ is an f-semi module homomorphism if and only if the following conditions are true:

(a) $\varphi(x+y)=\varphi(x)+\varphi(y)$ for all $x,y \in M$.

(b) There is a ring homomorphism $f: R \to T$ such $\varphi(r.x)=f(r).\varphi(x)$ for all $r \in R, x \in$M.

Remark 3.2 :

(a) The concept of semi homomorphism can be used to study relationships between two modules defined over different rings.

(b) It is easy to see that every homomorphism between two modules defined over the same ring is a semi homomorphism. (Semi homomorphisms generalize classical module homomorphisms).

In the following theorem, we show that every neutrosophic module M(I) defined over a neutrosophic ring R(I) is a semi homomorphic image to the corresponding refined neutrosophic module $M(I_1,I_2)$ over the corresponding refined neutrosophic ring $R(I_1,I_2)$.

Theorem 3.3 :

Let M be a module over a ring R, M(I) be the corresponding strong neutrosophic module over R(I), $M(I_1,I_2)$ be the corresponding strong refined neutrosophic module over $R(I_1,I_2)$. Then $\varphi: M(I_1,I_2) \to M(I)$; $\varphi(a,bI_1,cI_2)=a+(b+c)I$ is a semi homomorphism.

Proof:

Clearly, φ is a well defined map. Let $x = (a, bI_1, cI_2), y = (u, vI_1, wI_2)$ be two arbitrary elements in $M(I_1, I_2)$,

$\varphi(x + y) = (a + u) + (b + v + c + w)I = [a + (b + c)I] + [u + (v + w)I] = \varphi(x) + \varphi(y)$.

According to [3], the map $f: R(I_1, I_2) \to R(I); f(m, nI_1, tI_2) = m + (n + t)I; m, n, t \in R$ is a ring homomorphism.

Consider $r = (m, nI_1, tI_2) \in R(I_1, I_2)$, we can write

$\varphi(r.x) = \varphi(m.a, (m.b + n.a + n.b + n.c + t.b)I_1, (t.a + t.c + m.c)I_2) =$

$m.a + (m.b + n.a + n.b + n.c + t.b + t.a + t.c + m.c)I = [m + (n + t)I].[a + (b + c)I] =$

$f(r).\varphi(x)$. Thus the proof is complete.

Theorem 3.4 :

Let M be a module over a ring R, M(I) be the corresponding strong neutrosophic module over R(I), $M(I_1, I_2)$ be the corresponding strong refined neutrosophic module over $R(I_1, I_2), \varphi$ be the semi homomorphism defined in Theorem 3.3. Then

The semi homomorphic image of any strong submodule of $M(I_1, I_2)$ is a strong submodule of M(I).

Proof:

Let N be any strong AH-submodule of $M(I_1, I_2)$, since $(N, +)$ is a subgroup of $(M(I_1, I_2), +)$, we have $(\varphi(N), +)$ is a subgroup of (M(I),+).

Let $y = a + bI$ be an arbitrary element in $\varphi(N)$, $r = u + vI$ be any element in R(I), since φ, f are surjective maps, there are

$x = (a, mI_1, nI_2) \in M(I_1, I_2)$; $\varphi(x) = y$, i.e $m + n = b$ and $g = (u, zI_1, qI_2) \in R(I_1, I_2); f(g) = r$, i.e $z + q = v$,

$r.y = u.a + (v.a + v.b + u.b)I = u.a + (z.a + q.a + z.m + z.n + q.m + q.n + u.m + u.n)I=$

$f(g).\varphi(x) = \varphi(g.x) \in \varphi(N)$.

Now we study the kernel of semi homomorphism.

Definition 3.5:

Let M be a module over a ring R, M(I) be the corresponding strong neutrosophic module over R(I), $M(I_1, I_2)$ be the corresponding strong refined neutrosophic module over $R(I_1, I_2), \varphi$ be the semi homomorphism defined in Theorem 3.3. We define $Ker(\varphi) = \{x \in M(I_1, I_2);\ \varphi(x) = 0\}$.

Theorem 3.6:

Let M be a module over a ring R with unity 1, M(I) be the corresponding strong neutrosophic module over R(I), $M(I_1, I_2)$ be the corresponding strong refined neutrosophic module over $R(I_1, I_2), \varphi$ be the semi homomorphism defined in Theorem 3.3, we have

(a) $Ker(\varphi) = \{(0, xI_1, -xI_2); x \in M\}$.

(b) For each $m \in Ker(\varphi)$ there is $r \in Ker(f)$ and $n \in M(I_1, I_2)$ such $m = r.n$.

(c) $Ker(\varphi)$ is a strong submodule of $M(I_1, I_2)$.

Proof:

(a) Let $z = (y, xI_1, tI_2) \in Ker(\varphi)$, $\varphi(z) = y + (x + t)I = 0$, thus $y = 0, t = -x$.

(b) Consider $m = (0, xI_1, -xI_2) \in Ker(\varphi)$, there is $r = (0, I_1, -I_2) \in Ker(f) and\ n = (2x, xI_1, -xI_2) \in M(I_1, I_2); x \in M$, where

$r.n = (0, xI_1, -xI_2) = m$.

(c) It is clear that $Ker(\varphi)$ is closed under addition. Now suppose that $m = (0, xI_1, -xI_2) \in Ker(\varphi)$ and $r = (a, bI_1, cI_2) \in R(I_1, I_2)$, $r.m = (0, [a.x + c.x]I_1, [-a.x - c.x]I_2) \in Ker(\varphi)$.

Thus our proof is complete.

Example 3.7:

Let $M = Z_3$ be the module of integers modulo 3 over the ring Z, $M(I), M(I_1, I_2)$ be its corresponding neutrosophic and refined neutrosophic modules over $Z(I) and\ Z(I_1, I_2)$ respectively. We have

(a) $f: Z(I_1, I_2) \to Z(I); f(a, bI_1, cI_2) = a + (b + c)I$ is a ring homomorphism.

(b) $\varphi: M(I_1, I_2) \to M(I);\ \varphi(x, yI_1, zI_2) = x + (y + z)I$ is an f −semi module homomorphism.

(c) $Ker(\varphi) = \{(0, xI_1, -xI_2); x \in M\} = \{(0,0,0), (0, I_1, 2I_2), (0,2I_1, I_2)\}$.

Definition 3.8:

Let M be a module over a ring R with unity 1, M(I) be the corresponding strong neutrosophic module over R(I), $M(I_1, I_2)$ be the corresponding strong refined neutrosophic module over $R(I_1, I_2), \varphi$ be the semi homomorphism defined in Theorem 3.3. We define

$M(I_1,I_2)/Ker(\varphi) = \{a + Ker(\varphi); a \in M(I_1,I_2)\} = \{(x, yI_1, zI_2) + Ker(\varphi); x, y, z \in M\}$.

$M(I_1,I_2)/Ker(\varphi)$ is called neutrosophic semi factor.

Definition 3.9:

We define operations on the semi factor $M(I_1,I_2)/Ker(\varphi)$ as follows:

(a) Addition: for each $a + Ker(\varphi), b + Ker(\varphi) \in M(I_1,I_2)/Ker(\varphi)$, *we have*

$\left(a + Ker(\varphi)\right) + \left(b + Ker(\varphi)\right) = (a + b) + Ker(\varphi)$.

(b) Multiplication by a scalar: for each $r \in R(I_1,I_2), a + Ker(\varphi) \in M(I_1,I_2)/Ker(\varphi)$, we have

$r.\left(a + Ker(\varphi)\right) = r.a + Ker(\varphi)$.

Theorem 3.10:

Addition and Multiplication by a scalar are well defined operations on $M(I_1,I_2)/Ker(\varphi)$.

Proof:

Suppose that $a + Ker(\varphi) = b + Ker(\varphi)$, and $c + Ker(\varphi) = d + Ker(\varphi)$ then $a - b, c - d \in Ker(\varphi)$, thus $(a + c) - (b + d) \in Ker(\varphi)$, hence $a + c + Ker(\varphi) = b + d + Ker(\varphi)$.

Now assume that $a + Ker(\varphi) = b + Ker(\varphi)$ and $r = s \in R(I_1,I_2)$, then

$r.\left(a + Ker(\varphi)\right) = s.(b + Ker(\varphi))$. Thus the proof is complete.

Theorem 3.11:

$(M(I_1,I_2)/Ker(\varphi), +, .)$ is a module over the refined neutrosophic ring $R(I_1,I_2)$.

Proof:

Firstly, we remark that $M(I_1,I_2)/Ker(\varphi)$ is closed under addition and multiplication.

Since R is a ring with unity, we find that $1 \in R(I_1,I_2)$. Let $a + Ker(\varphi), b + Ker(\varphi)$ be two arbitrary elements in $M(I_1,I_2)/Ker(\varphi)$, r, s be two arbitrary elements in $R(I_1,I_2)$. Now we have

$1.\left(a + Ker(\varphi)\right) = a + Ker(\varphi)$, $(r + s)\left(a + Ker(\varphi)\right) = r.\left(a + Ker(\varphi)\right) + s.(b + Ker(\varphi))$,

$r.\left[\left(a + Ker(\varphi)\right) + \left(b + Ker(\varphi)\right)\right] = r.\left(a + b + Ker(\varphi)\right) = r.(a + b) + Ker(\varphi) = [r.a + Ker(\varphi)] + [r.b + Ker(\varphi)] = r.\left(a + Ker(\varphi)\right) + r.(b + Ker(\varphi))$.

Also, $M(I_1,I_2)/Ker(\varphi)$ is an abelian group with respect to addition. Thus it is a module over the ring $R(I_1,I_2)$.

Definition 3.12:

Let M be a module over a ring R, N be a module over a ring T, $\varphi: M \to N$ be a semi module homomorphism, we say that φ is a semi isomorphism if and only if it is a bijective map.

M, N are called semi isomorphic modules.

Theorem 3.13:

Let M be a module over a ring R with unity 1, M(I) be the corresponding strong neutrosophic module over R(I), $M(I_1, I_2)$ be the corresponding strong refined neutrosophic module over $R(I_1, I_2), \varphi$ be the semi homomorphism defined in Theorem 3.3. Then $M(I_1, I_2)/Ker(\varphi)$ is semi isomorphic to M(I).

Proof:

Define h: $M(I_1, I_2)/Ker(\varphi) \to M(I); h\big(a + Ker(\varphi)\big) = \varphi(a)$.

(a) h is well defined

Assume that $a + Ker(\varphi) = b + Ker(\varphi)$, then $a - b \in Ker(\varphi)$, hence $\varphi(a - b) = 0$, this means $\varphi(a) = \varphi(b)$.

(b) h is a semi homomorphism

Let $a + Ker(\varphi), b + Ker(\varphi)$ be two arbitrary elements in $M(I_1, I_2)/Ker(\varphi)$, and $r = (r_0, r_1 I_1,\ r_2 I_2)$ be an arbitrary element in $R(I_1, I_2)$, we have

$h([a + Ker(\varphi)] + [b + Ker(\varphi)]) = h\big(a + b + Ker(\varphi)\big) = \varphi(a + b)$=

$\varphi(a) + \varphi(b) = h(a + Ker(\varphi)) + h(b + Ker(\varphi))$.

$h(r.[a + Ker(\varphi)]) = h\big(r.a + Ker(\varphi)\big) = \varphi(r.a) = f(r).\varphi(a) = f(r).h(a + Ker(\varphi))$.

(c) h is a bijective map

It is easy to see that h is surjective. Now suppose that

$h([a + Ker(\varphi)]) = h([b + Ker(\varphi)])$, then $\varphi(a) = \varphi(b), \textit{this implies } \varphi(a - b) = 0$, hence

$a - b \in Ker(\varphi), \textit{so } a + Ker(\varphi) = b + Ker(\varphi)$. Thus h is a semi isomorphism.

Thus we get the proof.

Theorem 3.14:

Let $\varphi: M \to N, \tau: N \to L$ be two semi homomorphisms, where M, N, L are three modules over the rings

R, T, S respectively. Then $\tau o \varphi: M \to L$ is a semi homomorphism.

Proof:

Suppose that φ is an $f - semi\ homomorphism$, where $f: R \to T$ is a ring homomorphism, and τ is a $g - semi\ homomorphism$, where $g: T \to S$ is a ring homomorphism. It is clear that $gof: R \to S$ is a ring homomorphism. Now we prove that $\tau o \varphi: M \to L$ is a $gof - semi\ homomorphism$.

Let x, y be two arbitrary elements in M, and r be any element in the ring R, we have

$\tau o \varphi(x + y) = \tau\big(\varphi(x) + \varphi(y)\big) = \tau o \varphi(x) + \tau o \varphi(y)$.

$\tau o \varphi(r.x) = \tau\big(\varphi(r.x)\big) = \tau\big(f(r).\varphi(x)\big) = g\big(f(r)\big)\tau\big(\varphi(x)\big) = gof(r).\tau o \varphi(x)$. Hence

$\tau o \varphi$ is $gof - semi\ homomorphism$.

Remark 3.15:

The previous theorem shows that the set of all semi homomorphisms between a module M and itself is closed under multiplication.

The following theorem shows that any module M will be a semi homomorphic image to its corresponding neutrosophic module M(I).

Theorem 3.16:

Let M be a module over R, M(I) be its corresponding neutrosophic module over R(I).

Then M is a semi homomorphic image to M(I).

Proof:

According to [3], there is a ring homomorphism $f: R(I) \to R; f(r + sI) = r; r, s \in R$,

we define $g: M(I) \to M; g(x + yI) = x;\ x, y \in M$.

It is clear that for every $m, n \in M(I)$, we get $g(m + n) = g(m) + g(n)$. Also, we have $g([r + s.I].[x + yI]) = g(r.x + [r.y + s.x + s.y]I) = r.x = f(r + s.I).g(x + yI)$, thus g is a semi homomorphism.

The previous result shows that neutrosophic module (which they were defined using logic) have an algebraic origin, since they can be represented by semi homomorphisms.

Remark 3.17:

According to Theorem 3.13, we find that $^{\mathrm{M(I)}}/_{\mathrm{Kerg}}$ is semi isomorphic to M.

$Ker(g) = \{yI; y \in M\} = MI$. Hence $^{\mathrm{M(I)}}/_{\mathrm{MI}}$ is semi isomorphic to M.

Example 3.18:

Let $M = Z_6$ be a module over the ring of integers Z, M(I)= $\{x + yI; x, y \in Z_6\}$ be its corresponding neutrosophic module, we have

(a) $\ker(g) = MI = \{yI; y \in Z_6\} = \{0, I, 2I, 3I, 4I, 5I\}$.

(b) $^{\mathrm{M(I)}}/_{\mathrm{MI}} = \{A + MI; A \in M(I)\} = \{MI, 1 + MI, 2 + MI, 3 + MI, 4 + MI, 5 + MI\}$.

Which is semi isomorphic to M.

Definition 3.19:

Let R be any ring with unity, M be a module over R, we define

(a) The set of all semi homomorphisms from a module M to itself is denoted by

$S_M = \{\varphi: M \to M;$ φ is a semi homomorphism$\}$.

(b) The set of all semi isomorphisms from a module M to itself is denoted by $SI_M = \{\varphi: M \to M;\ \varphi\ is\ a\ semi\ isomorphism\}$.

(c) The set of all f −semi homomorphisms between a module M and itself is denoted by $f - S_M$.

(d) The set of all f −semi isomorphisms from a module M to itself is denoted by $f - SI_M$.

The following theorem clarifies the algebraic structure of semi homomorphisms.

Theorem 3.20:

Let M be a module over a ring R, $S_M, SI_M, f - S_M, f - SI_M$ be the sets defined above, we have:

(a) $(f - S_M, +, .)$ is a module over the ring R.

(b) (SI_M, o) is a semi group.

(c) If f is an isomorphism with property $fof = I$ (identity map), then for every $\varphi, \tau \in f - SI_M$, we have $\varphi o \tau$ is a module isomorphism.

Proof:

(a) Let g, h be any two $f - semi\ homomorphisms$. They are additive homomorphisms on M, thus $(g + h)(x + y) = (g + h)(x) + (g + h)(y)$ for every $x, y \in M$ clearly. Also, we have for every $r \in R\ and\ x \in M$: $(g+h)(r.x) = g(r.x) + h(r.x) = f(r).g(x) + f(r).h(x) = f(r).(g+h)(x)$, hence $g+h$ is an

$f - semi\ homomorphism$.

One the other hand, g has an inverse with respect to addition, which is $-g$, and $t(x) = 0$ is the identity in $f - S_M$, thus $(f - S_M, +)$ is an abelian group.

Now, let $a \in R$ be an arbitrary element, then the map $(a.g): M \to M; (a.g)(x) = a.g(x)$ is an $f - semi\ homomorphism$ clearly. It is easy to check that the rest of module's axioms are true.

(b) SI_M is closed under (o), and (o) is an associative operation, and the identity map $I \in SI_M$. thus the proof holds easily.

(c) It is well known that $\varphi o \tau$ is an additive homomorphism according to Theorem 3.14.

Now we shall prove that $\varphi o \tau\ (r.x) = r.\varphi o \tau$ (x), for every $r \in R\ and\ x \in M$.

$\varphi o \tau\ (r.x) = \varphi(\tau(r.x)) = \varphi[f(r).\tau(x)] = f(f(r)).\varphi o \tau\ (x) = I(r).\varphi o \tau\ (x) = r.\varphi o \tau\ (x)$. Thus $\varphi o \tau$ is a classical homomorphism.

4. Conclusions

In this article, we have defined the concept of semi homomorphism and semi isomorphism between two modules defined over different rings. Also, we applied this concept to study the algebraic relation between refined neutrosophic strong module and neutrosophic strong module, and between a module and its corresponding neutrosophic strong module.

The main result of this work is to prove that every strong neutrosophic module is a semi homomorphic image of the corresponding strong refined neutrosophic module.

In particular, we have constructed some examples to clarify the validity of our work.

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Foundations of Neutrosophic Number Theory

[1]Mohammad Abobala

[1]Faculty of science,, Department of Mathematics Tishreen University, Lattakia Syria

[1]e-mail: mohammadabobala777@gmail.com

Abstract: The aim of this paper is to establish a strong foundation of number theoretical concepts in the neutrosophic ring of integers $Z(I)$. This work generalizes and deals with necessary and sufficient conditions for division, Euler's function, congruencies, and some other classical concepts in $Z(I)$. The main result of this work is to show that Euler's famous theorem is still true in the case of neutrosophic integers. Also, this work introduces an algorithm to solve Pell's equation in the neutrosophic ring of integers Z(I).

Keywords: Neutrosophic Euler's theorem, neutrosophic integers, neutrosophic congruence, neutrosophic Pell's equation

1. Introduction

Neutrosophy is a new branch of philosophy founded by Smarandache to deal with indeterminacy in nature and science [12]. Neutrosophy has many important applications in many fields of knowledge such as computing [21], decision making [20], medical research [15], and applied science [22]. Then, it plays an important role in algebra, where many neutrosophic algebraic structures were defined and studied widely such as neutrosophic rings [1,8], neutrosophic vector spaces [4,14], neutrosophic modules [5,18], and refined neutrosophic rings [2,3,6,7,19]. Also, neutrosophy has many applications and effects on the progression of optimization [16], intelligent systems [13], and medical researches [15].

In the literature, number theory was a mathematical way to deal with the properties of integers such as Diophantine equations, primes, Euclidean division, and congruencies [10].

Neutrosophic number theory began in [9], where some properties of neutrosophic integers were introduced such as the form of primes in Z(I). Also, neutrosophic linear Diophantine equation was solved for the first time in [11].

This work is devoted to establish the theoretical foundations of neutrosophic number theory to deal with properties of neutrosophic integers. We aim to close an important research gap by determining algorithms and conditions for division, congruencies, neutrosophic Pell's equation, and Euler's function and theorem in Z(I).

Preliminaries

Definition 2.1: [1]

Let R be any ring, I be an indeterminacy with the property $I^2 = I$. Then $R(I) = \{a + bI; a, b \in R\}$ is called a neutrosophic ring.

If $R = Z$ is the ring of integers, then $Z(I) = \{a + bI; a, b \in Z\}$ is called the neutrosophic ring of integers. Elements of $Z(I)$ are called neutrosophic integers.

Remark: The notion of indeterminacy I was proposed by Smarandache and Kandasamy in [8] as an algebraic element instead of logical meaning. We deal with it by using its multiplicative property $I^2 = I$, which helps in the building of neutrosophic algebraic structures.

Definition 2.2: [10]

Pell's equation is the Diophantine equation with form $X^2 - DY^2 = N; D, N \in Z$.

Theorem 2.3: [10]

If the equation $X^2 - DY^2 = 1$ has a solution, then $D > 0$ *and* D is square free.

Theorem 2.4: [10]

$Z[\sqrt{d_1}]$ is an integral domain, where d_1 is a square free integer.

Theorem 2.5: [9]

Let Z(I)= $\{a + bI; a, b \in Z\}$ the neutrosophic ring of integers. Then primes in Z(I) have one of the following forms:

$x = \pm p + (\pm 1 \pm p)I \ or \ x = \pm 1 + (\pm p \pm 1)I; p \ is \ any \ prime \ in \ Z.$

Definition 2.6: [19]

Let $R(I) = \{a + bI;\ a, b \in R\}$ be the real neutrosophic field, we say that $a + bI \le c + dI$ if and only if $a \le c \ and \ a + b \le c + d$.

Theorem 2.7: [19]

The relation defined in Definition 2.6 is an order relation.

Remark 2.8: [19]

According to Theorem 2.7, we are able to define positive neutrosophic real numbers as follows:

$a + bI \geq 0 = 0 + 0.I$ implies that $a \geq 0, a + b \geq 0$.

Absolute value on R(I) can be defined as follows:

$|a + bI| = |a| + I[|a + b| - |a|]$, we can see that $|a + bI| \geq 0$.

Example 2.9: [19]

$x = 2 - I$ is a neutrosophic positive real number, since $2 \geq 0$ *and* $(2 - 1) = 1 \geq 0$.

$2 + I \geq 2$, that is because $2 \geq 2$ *and* $(2 + 1) = 3 \geq (2 + 0) = 2$.

3. Number Theory in $\boldsymbol{Z(I)}$

Definition 3.1: (Division)

Let $Z(I) = \{a + bI; a, b \in Z\}$ the neutrosophic ring of integers. For any $x, y \in Z(I)$, we say that $x|y$ if there is $r \in Z(I)$; $r.x = y$.

Theorem 3.2: (Form of division in *Z(I))*

Let Z(I)= $\{a + bI; a, b \in Z\}$ the neutrosophic ring of integers, $x = x_1 + x_2I, y = y_1 + y_2I$ be two arbitrary elements in Z(I). Then $x|y$ if and only if $x_1|y_1$ *and* $x_1 + x_2|y_1 + y_2$.

Proof:

Suppose that $x|y$, hence there is $r = r_1 + r_2I \in Z(I)$; $r.x = y$. This implies

(I) $r_1x_1 = y_1$, i.e. $x_1|y_1$.

(II) $r_1x_2 + r_2x_1 + r_2x_2 = y_2$. By adding (I) to (II) we get

$r_1x_1 + r_1x_2 + r_2x_1 + r_2x_2 = y_1 + y_2$, this means that $(r_1 + r_2)(x_1 + x_2) = y_1 + y_2$.

Thus $x_1 + x_2|y_1 + y_2$.

Conversely, assume that $x_1|y_1$ *and* $x_1 + x_2|y_1 + y_2$, hence there is $a, b \in Z$ such that $ax_1 = y_1$ *and* $b(x_1 + x_2) = y_1 + y_2$. We put $r = a + (b - a)I$.

It is easy to see that $r.x = y$ and $x|y$.

Definition 3.3: (primes)

Let Z(I)= $\{a+bI; a,b \in Z\}$ the neutrosophic ring of integers. An arbitrary element $x \in Z(I)$ is called prime if $x|y.z$ implies $x|y \; or \; x|z$.

Theorem 3.4: (Form of primes in Z(I))

This result was proved in [9].

Let Z(I)= $\{a+bI; a,b \in Z\}$ the neutrosophic ring of integers**.** Then primes in Z(I) have one of the following forms:

$x = \pm p + (\pm 1 \pm p)I \; or \; x = \pm 1 + (\pm p \pm 1)I; p \; is \; any \; prime \; in \; Z.$

Definition 3.5: (Congruence)

(a) Let $x = a + bI, y = c + dI, z = m + nI$ be three elements in Z(I). We say that $x \equiv y(mod z)$ if and only if $z|x - y$.

(b) We say that $z = \gcd(x,y)$ if and only if $z|x \; and \; z|y$ and for each divisor $c|x \; and \; c|y$, then $c|z$.

x, y are called relatively prime in Z(I) if and only if $\gcd(x,y) = 1$.

Theorem 3.6: (Form of congruencies in Z(I))

Let $x = a + bI, y = c + dI, z = m + nI$ be three elements in Z(I). Then $x \equiv y(mod z)$ if and only if $a \equiv c(mod \; m), a + b \equiv c + d(mod \; m + n)$.

Proof:

We suppose that $x \equiv y(mod z)$, hence $z|x - y$, i.e. $m + nI|(a - c) + (b - d)I$. This implies $m|a - c \; and \; m + n|(a + b) - (c + d)$, thus $a \equiv c(mod \; m), a + b \equiv c + d(mod \; m + n)$.

Conversely, we suppose that $a \equiv c(mod \; m), a + b \equiv c + d(mod \; m + n)$, hence $m|a - c \; and \; m + n|(a + b) - (c + d)$, this implies that $m + nI|(a - c) + (b - d)I$, i.e. $z|x - y$, which means that $x \equiv y(mod z)$.

Theorem 3.7:

Let $x = a + bI, y = c + dI, z = m + nI$ be three elements in *Z(I)*. Then $z = \gcd(x,y)$ if $m = \gcd(a,c) \; and \; m + n = \gcd(a + b, c + d)$.

Proof:

Consider $z = m + nI$, where $m = \gcd(a,c) \; and \; m + n = \gcd(a + b, c + d)$.

It is easy to check that $z|x \; and \; z|y$, that is because $m = \gcd(a,c) \,|a, m = \gcd(a,c) \,|c$, and

$m+n=\gcd(a+b,c+d)\,|a+b, m+n=\gcd(a+b,c+d)|c+d$. On the other hand, we assume that $l=f+gI$ is a common divisor of x and y. We shall prove that $l|z$.

Since l is a common divisor, then we have $f|a\ and\ f|c$, hence $f|\gcd(a,c)=m$. Also, we have $f+g|a+b\ and\ f+g|c+d$, hence $f+g|\gcd(a+b,c+d)=m+n$. This implies that $l|z$, and $z=\gcd(x,y)$.

Example 3.8:

(a) $3+5I\equiv(1+3I)(mod\ 2+2I)$. This is because $3\equiv 1(mod\ 2), 3+5=8\equiv 1+3=4(mod 4)$.

(b) $\gcd(3+5I,1+3I)=1+3I$, that is because $\gcd(3,1)=1=m, \gcd(3+5,1+3)=\gcd(8,4)=4=m+n$, thus $m+nI=1+3I=\gcd(3+5I,1+3I)$.

Theorem 3.9: (Euclidian division theorem in Z(I))

Let Z(I) be the neutrosophic ring of integers, $x=a+bI, y=c+dI$ be two arbitrary elements in Z(I). There are two elements $q=s+tI, r=m+nI$ such that $x=q.y+r$.

Proof:

This proof is different from the proof which was introduced in [9].

By the division theorem in Z, we can find the following integers:

q_1,q_2,r_1,r_2: $a=q_1c+r_1, and\ a+b=(c+d)q_2+r_2$. By putting $s=q_1, t=(q_2-q_1), m=r_1, n=(r_2-r_1)$, we find that $x=q.y+r$.

Example 3.10:

Consider the following neutrosophic integers $x=5+4I, y=3+I$. There are $q=1+I, r=2-I$ such that $x=q.y+r$.

Remark 3.11: (Solvability of a linear congruence in Z(I))

To solve a linear congruence $x+yI\equiv a+bI(mod\ m+nI)$. We should take its equivalent congruencies according to Theorem 3.6:

$x\equiv a(mod\ m), and\ x+y\equiv(a+b)(mod\ m+n)$. We solve the equivalent system, and compute x,y.

Example 3.12:

Consider the following neutrosophic linear congruence (*) $x+yI\equiv 1+7I(mod\ 4+I)$. Its equivalent system is:

(a) $x\equiv 1(mod\ 4)$. (It has a solution $x=1$).

$(b) x+y \equiv 8(mod\ 5)$. (It has a solution $x+y=3, hence\ y=2$. This means that $1+2I$ is a solution of the neutrosophic congruence (*).

We can see that $4+I|(1+2I)-(1+7I)$, that is because $(4+I)(-I)=-5I$.

Definition 3.14: (Euler's function in Z(I))

We define the neutrosophic Euler's function on Z(I) as follows:

$\varphi(a+bI)=|\{x=c+dI; \gcd(c+dI, a+bI)=1\}|, where\ c+dI \leq a+bI$.

Theorem 3.15: (Euler's Theorem in *Z(I))*

(a) Let $x=a+bI$ be any element in *Z(I),* then $\varphi(x)=\varphi(a)\times\varphi(b+a)$.

(b) If $y=c+dI$ is a neutrosophic integer with $\gcd(x,y)=1$, hence $y^{\varphi(x)} \equiv 1(mod\ x)$.

(neutrosophic Euler's Theorem).

Proof:

(a) Let $y=c+dI$ be any neutrosophic integer with, $c+dI \leq a+bI, and$ $\gcd(x,y)=1$. We can see by Theorem 3.7 that

$\gcd(a,c)=1, \gcd(a+b, c+d)=1$, i.e. (a,c) are relatively prime and $(a+b, c+d)$ are relatively prime, hence we get that $\varphi(x)=\varphi(a)\times\varphi(b+a)$.

(b) By classical Euler's Theorem, we have $c^{\varphi(a)} \equiv 1(mod\ a), and\ (c+d)^{\varphi(a+b)} \equiv 1(mod\ a+b)$, that is because $\gcd(a,c)=\gcd(a+b,c+d)=1$ under the assumption of $\gcd(x,y)=1$. Now, we can write $c^{\varphi(a)\times\varphi(b+a)}=c^{\varphi(x)} \equiv 1(mod\ a)$, $(c+d)^{\varphi(a)\times\varphi(b+a)}=(c+d)^{\varphi(x)} \equiv 1(mod\ a+b)$.

Now, we compute

$$y^{\varphi(x)}=(c+dI)^{\varphi(x)}=c^{\varphi(x)}+I[\sum_{i=1}^{\varphi(x)}\binom{\varphi(x)}{i}c^{\varphi(x)-i}d^i]=c^{\varphi(x)}+I\left[(c+d)^{\varphi(x)}-c^{\varphi(x)}\right]=m+nI.$$

We remark that $m=c^{\varphi(x)} \equiv 1(mod\ a), m+n=(c+d)^{\varphi(x)} \equiv 1(mod\ a+b)$, this implies that $y^{\varphi(x)}=m+nI \equiv 1(mod\ a+bI)$, according to Theorem 3.6.

The previous theorem will open a new door in the study of neutrosophic number theory, since it clarifies that Euler's famous theorem is still true in the case of neutrosophic integers.

Remark 3.16: (Solving a congruence linear system in Z(I))

To solve a linear system of congruencies in Z(I), we can solve the corresponding equivalent system in Z.

Example 3.17:

Consider the following linear system of congruencies in Z(I).

$2x+(3y-2x)I \equiv 3+I(mod\ 7+4I), 4x+(y-4x)I \equiv 7-5I(mod\ 13-10I)$, we aim to find x, y.

The corresponding linear system in Z according to Theorem 3.6 is

$2x \equiv 3(mod 7), 3y \equiv 4(mod\ 11), 4x \equiv 7(mod\ 13), y \equiv 2(mod 3)$, it has a solution $x=y=5$.

Thus the neutrosophic congruence in Z(I) has a solution $10+5I, 20-15I$.

4. Neutrosophic Pell's equation

Definition 4.1:

Let $Z(I)=\{a+bI;\ a,b \in Z\}$ be the neutrosophic ring of integers. The neutrosophic Pell's Equation in Z(I) is defined as follows:

$X^2-DY^2=C;\ X,Y,D,C \in Z(I)$.

We show the sufficient condition for solvability of neutrosophic Pell's equation.

Theorem 4.2:

Let $Z(I)=\{a+bI;\ a,b \in Z\}$ be the neutrosophic ring of integers,(*) $X^2-DY^2=C;\ X,Y,D,C \in Z(I)$ be a neutrosophic Pell's equation with $X=x_1+x_2I, Y=y_1+y_2I, D=d_1+d_2I, C=c_1+c_2I$. This equation is equivalent to the following two classical Pell's equations:

(a) ${x_1}^2-d_1{y_1}^2=c_1$.
(b) $(x_1+x_2)^2-(d_1+d_2)(y_1+y_2)^2=c_1+c_2$.
Proof:

It is sufficient to prove that equation (*) implies (a), (b).

By computing (*), we get

$(x_1+x_2I)^2-(d_1+d_2I)(y_1+y_2I)^2=c_1+c_2I$, this implies

$[{x_1}^2-d_1{y_1}^2]+I[2x_1x_2+{x_2}^2-d_1{y_2}^2-d_2{y_1}^2-2d_1y_1y_2-2d_2y_1y_2-d_2{y_1}^2-d_2{y_2}^2-2d_1d_2{y_1}^2-2d_1d_2y_1y_2-2d_1d_2{y_2}^2]=c_1+c_2I$, thus

${x_1}^2-d_1{y_1}^2=c_1$. (Equation (a)), and

(**) $2x_1x_2+{x_2}^2-d_1{y_2}^2-d_2{y_1}^2-2d_1y_1y_2-2d_2y_1y_2-d_2{y_1}^2-d_2{y_2}^2-2d_1d_2{y_1}^2-2d_1d_2y_1y_2-2d_1d_2{y_2}^2=c_2$, by adding equation (a) to (**), we get

${x_1}^2-d_1{y_1}^2+2x_1x_2+{x_2}^2-d_1{y_2}^2-d_2{y_1}^2-2d_1y_1y_2-2d_2y_1y_2-d_2{y_1}^2-d_2{y_2}^2-2d_1d_2{y_1}^2-2d_1d_2y_1y_2-2d_1d_2{y_2}^2=c_1+c_2$, hence

$(x_1+x_2)^2-(d_1+d_2)(y_1+y_2)^2=c_1+c_2$. (Equation (b)).

Remark 4.3:

To solve the neutrosophic Pell's equation $X^2 - DY^2 = C$, follow these steps

1) Solve $x_1{}^2 - d_1 y_1{}^2 = c_1$, if it is possible.

2) Solve $(x_1 + x_2)^2 - (d_1 + d_2)(y_1 + y_2)^2 = c_1 + c_2$, if it is possible.

3) Compute x_2, y_2.

We study some special neutrosophic Pell's equations.

Theorem 4.4:

If the neutrosophic Pell's equation $X^2 - DY^2 = 1$ has non trivial solutions, then

$d_1 > 0, d_1 + d_2 > 0$, and $d_1, d_1 + d_2$ are square free.

Proof:

According to Theorem 4.2, the equation $X^2 - DY^2 = 1$ is equivalent to
(a) $x_1{}^2 - d_1 y_1{}^2 = 1$.

(b) $(x_1 + x_2)^2 - (d_1 + d_2)(y_1 + y_2)^2 = 1$.

By Theorem , thus (a), (b) have non trivial solutions. By Theorem 2.3 , we find that $d_1 > 0, d_1 + d_2 > 0$, and $d_1, d_1 + d_2$ are square free.

Example 4.5:

The equation $X^2 - (2 + 3I)Y^2 = 1$ has non trivial solution, that is because:

The equivalent system is: (a) $x_1{}^2 - 2y_1{}^2 = 1$, (b) $(x_1 + x_2)^2 - 5(y_1 + y_2)^2 = 1$.

Equation (a) has a solution $x_1 = 3, y_1 = 2$. Equation (b) has a solution $x_1 + x_2 = 9, y_1 + y_2 = 4$, thus

$x_2 = 9 - x_1 = 6, y_2 = 4 - y_1 = 2$. So $X = 3 + 6I, Y = 2 + 2I$. We can see easily that $2 > 0, 2 + 3 = 5 > 0$, *and* $2, 2 + 3 = 5$ are square free.

Example 4.6:

Let $X^2 - (3 - I)Y^2 = -3 + I$ be a neutrosophic Pell's equation. Its equivalent system is

$x_1{}^2 - 3y_1{}^2 = -3, (x_1 + x_2)^2 - (2)(y_1 + y_2)^2 = -2$. The first equation has the solution

$x_1 = 3, y_1 = 2$, the second one has the solution

$x_1 + x_2 = 4, y_1 + y_2 = 3$, thus $x_2 = 1, y_2 = 1$. We find that $X = 3 + I, Y = 2 + I$ is a solution of $X^2 - (3 - I)Y^2 = -3 + I$.

Theorem 4.7:

If the Pell's equation $x_1{}^2 - d_1 y_1{}^2 = c_1;\ d_1, c_1 \in Z$ has m solutions exactly. Then the neutrosophic Pell's equation

$X^2 - d_1 Y^2 = c_1; X = x_1 + x_2 I, Y = y_1 + y_2 I$ has exactly m^2 solutions.

Proof:

$X^2 - d_1 Y^2 = c_1$ is equivalent to the system:

(a) $x_1{}^2 - d_1 y_1{}^2 = c_1$.

(b) $(x_1 + x_2)^2 - d_1(y_1 + y_2)^2 = c_1$.

We can see that (a), (b) are the same Pell's equation, thus each one has m solutions. Hence we have for each value of x_1, *(m)* corresponding values of x_2, and we get the same thing for y_1, y_2. Thus we have exactly m^2 solutions for equation $X^2 - d_1 Y^2 = c_1$.

Theorem 4.8:

If the neutrosophic Pell's equation $X^2 - Dy^2 = C;\ D = a - aI\ ; a \in Z$ is solvable, then $c_1 + c_2$ is a square.

Proof:

Suppose that $X^2 - Dy^2 = C$ has a solution $X = x_1 + x_2 I, Y = y_1 + y_2 I$, then

$x_1{}^2 - a y_1{}^2 = c_1,\ (x_1 + x_2)^2 - (a - a)(y_1 + y_2)^2 = c_1 + c_2$ are solvable equations, thus

$(x_1 + x_2)^2 = c_1 + c_2$, and $c_1 + c_2$ is a square.

Theorem 4.9:

If the neutrosophic Pell's equation $X^2 - Dy^2 = C; D = aI; a \in Z$ is solvable, then c_1 is a square.

Proof:

Suppose that $X^2 - Dy^2 = C$ has a solution $X = x_1 + x_2 I, Y = y_1 + y_2 I$, then

$x_1{}^2 - 0.y_1{}^2 = c_1,\ (x_1 + x_2)^2 - (a)(y_1 + y_2)^2 = c_1 + c_2$ are solvable equations, thus

$x_1{}^2 = c_1$, and c_1 is a square.

Remark 4.10:

If the neutrosophic Pell's equation $X^2 - Dy^2 = C; D = aI; a \in Z$ is solvable, then it has an infinite number of solutions. This is because $x_1 = \pm\sqrt{c_1}$ and $(y_1 + y_2)^2$ is constant, i.e there is an infinite number of possible solutions. For every value of y_1, there is a single related value of y_2.

Example 4.11:

Consider the following neutrosophic Pell's equation $X^2 - IY^2 = 1 + 4I$, the equivalent system is

$x_1{}^2 = 1,\ (x_1 + x_2)^2 - (y_1 + y_2)^2 = 5$. It has a solution $x_1 = 1, x_2 = 2, y_1 + y_2 = 2$.

We can see that the solutions of $X^2 - IY^2 = 1 + 4I$ are:

$X = 1 + 2I \text{ or } X = -1 + 4I, Y = y_1 + (2 - y_1)I.$

Theorem 4.12:

Let $x_1{}^2 - d_1y_1{}^2 = c_1, x_2{}^2 - d_2y_2{}^2 = c_2$ be two classical Pell's equations. They can be transformed into one corresponding neutrosophic Pell's equation (*) $X^2 - DY^2 = C;\ X = x_1 + (x_2 - x_1)I, Y = y_1 + (y_2 - y_1)I,$

$D = d_1 + (d_2 - d_1)I, C = c_1 + (c_2 - c_1)I.$

Proof:

The proof holds directly by easy computing of equation (*).

Example 4.13:

Let $x_1{}^2 - 2y_1{}^2 = 1, x_2{}^2 - 3y_2{}^2 = 5$ be two Pell's equations. The corresponding neutrosophic Pell's equation is $[x_1 + (x_2 - x_1)I]^2 - (2 + I)[y_1 + (y_2 - y_1)I]^2 = 1 + 4I.$

Theorem 4.14:

The neutrosophic Pell's equation (*) $X^2 - DY^2 = aI$; (d_1 is a positive integer and square free) has solutions if and only if the equation

$x_2{}^2 - (d_1 + d_2)y_2{}^2 = a$ has solutions. Its solution has the form $X = x_2I, Y = y_2I$.

Proof:

The equivalent system of (*) is:

(a) $x_1{}^2 - d_1y_1{}^2 = 0.$

(b) $(x_1 + x_2)^2 - (d_1 + d_2)(y_1 + y_2)^2 = a.$

Equation (a) has only the zero solution, that is because $Z[\sqrt{d_1}]$ is an integral domain, thus $x_1 = y_1 = 0$.

Equation (b) becomes $x_2{}^2 - (d_1 + d_2)y_2{}^2 = a$. Hence (*) has solutions if and only if (b) has solutions.

The solutions of (*) have the property $x_1 = y_1 = 0$, so they have the form $X = x_2 I, Y = y_2 I$.

4. Conclusions

In this article, we have established the basic theory of neutrosophic numbers. Concepts such as division, relatively primes, congruencies, and Pell's equation were discussed and handled in the case of neutrosophic integers. Also, we have proved that Euler's famous theorem is still true in *Z(I).*

This work can be considered as a primary step in the study of neutrosophic number theory, we aim that it will be very effective in the study of neutrosophic integers.

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NeutroOrderedAlgebra: Applications to Semigroups

Madeleine Al-Tahan[1,*], F. Smarandache[2], and Bijan Davvaz[3]
[1]Department of Mathematics, Lebanese International University, Bekaa, Lebanon; madeline.tahan@liu.edu.lb
[2]Mathematics and Science Division, University of New Mexico, Gallup Campus, USA, smarand@unm.edu
[3]Department of Mathematics, Yazd University, Yazd, Iran; davvaz@yazd.ac.ir
[*]Correspondence: madeline.tahan@liu.edu.lb

Abstract. Starting with a partial order on a NeutroAlgebra, we get a NeutroStructure. The latter if it satisfies the conditions of NeutroOrder, it becomes a NeutroOrderedAlgebra. In this paper, we apply our new defined notion to semigroups by studying NeutroOrderedSemigroups. More precisely, we define some related terms like NeutrosOrderedSemigroup, NeutroOrderedIdeal, NeutroOrderedFilter, NeutroOrderedHomomorphism, etc., illustrate them via some examples, and study some of their properties.

Keywords: NeutroAlgebra, NeutroSemigroup, NeutroOrderedAlgebra, NeutrosOrderedSemigroup, NeutroOrderedIdeal, NeutroOrderedFilter, NeutroOrderedHomomorphism, NeutroOrderedStrongHomomorphism.
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1. Introduction

Neutrosophy, the study of neutralities, is a new branch of Philosophy initiated by Smarandache in 1995. It has many applications in almost every field. Many algebraists worked on the connection between neutrosophy and algebraic structures. Fore more details, we refer to [1–3]. Unlike the idealistic or abstract algebraic structures, from pure mathematics, constructed on a given perfect space (set), where the axioms (laws, rules, theorems, results etc.) are totally (100%) true for all spaces elements, our world and reality consist of approximations, imperfections, vagueness, and partialities. Starting from the latter idea, Smarandache introduced NeutroAlgebra. In 2019 and 2020, he [11–13] generalized the classical Algebraic Structures to NeutroAlgebraic Structures (or NeutroAlgebras) whose operations and axioms are partially true, partially indeterminate, and partially false as extensions of Partial Algebra, and to AntiAlgebraic Structures (or AntiAlgebras) whose operations and axioms are totally false. And in general, he extended any classical Structure, in no matter what field of knowledge, to a

NeutroStructure and an AntiStructure. A Partial Algebra is an algebra that has at least one Partial Operation, and all its Axioms are classical. Through a theorem, Smarandache [11] proved that a NeutroAlgebra is a generalization of Partial Algebra and gave some examples of NeutroAlgebras that are not Partial Algebras. Many researchers worked on special types of NeutroAlgebras and AntiAlgebras by applying them to different types of algebraic structures such as groups, rings, BE-Algebras, BCK-Algebras, etc. For more details, we refer to [4–6, 9, 10, 14, 15].

Inspired by NeutroAlgebra and ordered Algebra, our paper introduces and studies NeutroOrderedAlgebra. And it is constructed as follows: After an Introduction, in Section 2, we introduce NeutroOrderedAlgebra and some related terms such as NeutroOrderedSubAlgebra and NeutroOrderedHomomorphism. And in Section 3, we apply the concept of NeutroOrderedAlgebra to semigroups and study NeutroOrderedSemigroups by presenting several examples and studying some of their interesting properties.

2. NeutroOrderedAlgebra

In this section, we combine the notions of ordered algebraic structures and NeutroAlgebra to introduce *NeutroOrderedAlgebra*. Some new definitions related to the new concept are presented. For details about ordered algebraic structures, we refer to [7, 8].

Definition 2.1. [11] A non-empty set A endowed with n operations "$\star_i$" for $i = 1, \dots, n$, is called *NeutroAlgebra* if it has at least one NeutroOperation or at least one NeutroAxiom with no AntiOperations nor AntiAxioms.

Definition 2.2. [8] Let A be an Algebra with n operations "$\star_i$" and "$\leq$" be a partial order (reflexive, anti-symmetric, and transitive) on A. Then $(A, \star_1, \dots, \star_n, \leq)$ is an Ordered Algebra if the following conditions hold.
If $x \leq y \in A$ then $z \star_i x \leq z \star_i y$ and $x \star_i z \leq y \star_i z$ for all $i = 1, \dots, n$ and $z \in A$.

Definition 2.3. Let A be a NeutroAlgebra with n (Neutro) operations "$\star_i$" and "$\leq$" be a partial order (reflexive, anti-symmetric, and transitive) on A. Then $(A, \star_1, \dots, \star_n, \leq)$ is a *NeutroOrderedAlgebra* if the following conditions hold.

(1) There exist $x \leq y \in A$ with $x \neq y$ such that $z \star_i x \leq z \star_i y$ and $x \star_i z \leq y \star_i z$ for all $z \in A$ and $i = 1, \dots, n$. (This condition is called degree of truth, "T".)
(2) There exist $x \leq y \in A$ and $z \in A$ such that $z \star_i x \nleq z \star_i y$ or $x \star_i z \nleq y \star_i z$ for some $i = 1, \dots, n$. (This condition is called degree of falsity, "F".)
(3) There exist $x \leq y \in A$ and $z \in A$ such that $z \star_i x$ or $z \star_i y$ or $x \star_i z$ or $y \star_i z$ are indeterminate, or the relation between $z \star_i x$ and $z \star_i y$, or the relation between $x \star_i z$

and $y \star_i z$ are indeterminate for some $i = 1, \ldots, n$. (This condition is called degree of indeterminacy, "I".)

Where (T, I, F) is different from $(1, 0, 0)$ that represents the classical Ordered Algebra as well from $(0, 0, 1)$ that represents the AntiOrderedAlgebra.

Definition 2.4. Let $(A, \star_1, \ldots, \star_n, \leq)$ be a NeutroOrderedAlgebra. If "$\leq$" is a total order on A then A is called *NeutroTotalOrderedAlgebra*.

Definition 2.5. Let $(A, \star_1, \ldots, \star_n, \leq_A)$ be a NeutroOrderedAlgebra and $\emptyset \neq S \subseteq A$. Then S is a *NeutroOrderedSubAlgebra* of A if $(S, \star_1, \ldots, \star_n, \leq_A)$ is a NeutroOrderedAlgebra and there exists $x \in S$ with $(x] = \{y \in A : y \leq_A x\} \subseteq S$.

Remark 2.6. A NeutroOrderedAlgebra has at least one NeutroOrderedSubAlgebra which is itself.

Definition 2.7. Let $(A, \star_1, \ldots, \star_n, \leq_A)$ and $(B, \circledast_1, \ldots, \circledast_n, \leq_B)$ be NeutroOrderedAlgebras and $\phi : A \to B$ be a function. Then

(1) ϕ is called *NeutroOrderedHomomorphism* if there exist $x, y \in A$ such that for all $i = 1, \ldots, n$, $\phi(x \star_i y) = \phi(x) \circledast_i \phi(y)$, and there exist $a \leq_A b \in A$ with $a \neq b$ such that $\phi(a) \leq_B \phi(b)$.

(2) ϕ is called *NeutroOrderedIsomomorphism* if ϕ is a bijective NeutroOrderedHomomorphism. In this case, we write $A \cong_I B$.

(3) ϕ is called *NeutroOrderedStrongHomomorphism* if for all $x, y \in A$ and for all $i = 1, \ldots, n$, we have $\phi(x \star_i y) = \phi(x) \circledast_i \phi(y)$ and $a \leq_A b \in A$ is equivalent to $\phi(a) \leq_B \phi(b)$ for all $a, b \in A$.

(4) ϕ is called *NeutroOrderedStrongIsomomorphism* if ϕ is a bijective NeutroOrderedStrongHomomorphism. In this case, we write $A \cong_{SI} B$.

Example 2.8. Let $(A, \star_1, \ldots, \star_n, \leq_A)$ be a NeutroOrderedAlgebra, B a NeutroOrderedSubAlgebra of A, and $\phi : B \to A$ be the inclusion map ($\phi(x) = x$ for all $x \in B$). Then ϕ is a NeutroOrderedStrongHomomorphism.

Example 2.9. Let $(A, \star_1, \ldots, \star_n, \leq_A)$ be a NeutroOrderedAlgebra and $\phi : A \to A$ be the identity map ($\phi(x) = x$ for all $x \in A$). Then ϕ is a NeutroOrderedStrongIsomomorphism.

Remark 2.10. Every NeutroOrderedStrongHomomorphism (NeutroOrderedStrongIsomorphism) is a NeutroOrderedHomomorphism (NeutroOrderedIsomorphism).

Theorem 2.11. *The relation "$\cong_{SI}$" is an equivalence relation on the set of NeutroOrderedAlgebras.*

Proof. By taking the identity map and using Example 2.9, we can easily prove that "$\cong_{SI}$" is a reflexive relation. Let $A \cong_{SI} B$. Then there exist a NeutroOrderedStrongIsomorphism $\phi : (A, \star_1, \ldots, \star_n, \leq_A) \to (B, \circledast_1, \ldots, \circledast_n, \leq_B)$. We prove that $\phi^{-1} : B \to A$ is a NeutroOrderedStrongIsomorphism. For all $b_1, b_2 \in B$, there exist $a_1, a_2 \in A$ with $\phi(a_1) = b_1$ and $\phi(a_2) = b_2$. For all $i = 1, \ldots, n$, we have:

$$\phi^{-1}(b_1 \circledast_i b_2) = \phi^{-1}(\phi(a_1) \circledast_i \phi(a_2)) = \phi^{-1}(\phi(a_1 \star_i a_2)) = a_1 \star_i a_2 = \phi^{-1}(b_1) \star_i \phi^{-1}(b_2).$$

Moreover, having $a_1 \leq_A a_2 \in A$ equivalent to $\phi(a_1) \leq_B \phi(a_2) \in B$ and ϕ an onto function implies that $b_1 = \phi(a_1) \leq_B \phi(a_2) = b_2 \in B$ is equivalent to $a_1 = \phi^{-1}(b_1) \leq_A a_2 = \phi^{-1}(b_2) \in A$. Thus, $B \cong_{SI} A$ and hence, "$\cong_{SI}$" is a symmetric relation. Let $A \cong_{SI} B$ and $B \cong_{SI} C$. Then there exist NeutroOrderedStrongIsomorphisms $\phi : A \to B$ and $\psi : B \to C$. One can easily see that $\psi \circ \phi : A \to C$ is a NeutroOrderedStrongIsomorphism. Thus, $A \cong_{SI} C$ and hence, "$\cong_{SI}$" is a transitive relation. □

Remark 2.12. The relation "$\cong_I$" is a reflexive and symmetric relation on the set of NeutroOrderedAlgebras. But it may fail to be a transitive relation.

3. **NeutroOrderedSemigroup**

In this section, we use the defined notion of NeutroOrderedAlgebra in Section 2 and apply it to semigroups. As a result, we define NeutroOrderedSemigroup and other related concepts. Moreover, we present some examples of finite as well as infinite NeutroOrderedSemigroups. Finally, we study some properties of NeutroOrderedSubSemigroups, NeutroOrderedIdeals, and NeutroOrderedFilters.

Definition 3.1. [8] Let $(S, \cdot)$ be a semigroup ("$\cdot$" is an associative and a binary closed operation) and "$\leq$" a partial order on S. Then $(S, \cdot, \leq)$ is an *ordered semigroup* if for every $x \leq y \in S$, $z \cdot x \leq z \cdot y$ and $x \cdot z \leq y \cdot z$ for all $z \in S$.

Definition 3.2. [8] Let $(S, \cdot, \leq)$ be an ordered semigroup and $\emptyset \neq M \subseteq S$. Then

(1) M is an *ordered subsemigroup* of S if $(M, \cdot, \leq)$ is an ordered semigroup and $(x] \subseteq M$ for all $x \in M$. i.e., if $y \leq x$ then $y \in M$.
(2) M is an *ordered left ideal* of S if M is an ordered subsemigroup of S and for all $x \in M$, $r \in S$, we have $rx \in M$.
(3) M is an *ordered right ideal* of S if M is an ordered subsemigroup of S and for all $x \in M$, $r \in S$, we have $xr \in M$.
(4) M is an *ordered ideal* of S if M is both: an ordered left ideal of S and an ordered right ideal of S.

(5) M is an *ordered filter* of S if $(M, \cdot)$ is a semigroup and for all $x, y \in S$ with $x \cdot y \in M$, we have $x, y \in M$ and $[y) \subseteq M$ for all $y \in M$. i.e., if $y \in M$ with $y \leq x$ then $x \in M$.

Definition 3.3. Let $(S, \cdot)$ be a NeutroSemigroup and "$\leq$" be a partial order (reflexive, anti-symmetric, and transitive) on S. Then $(S, \cdot, \leq)$ is a *NeutroOrderedSemigroup* if the following conditions hold.

(1) There exist $x \leq y \in S$ with $x \neq y$ such that $z \cdot x \leq z \cdot y$ and $x \cdot z \leq y \cdot z$ for all $z \in S$. (This condition is called degree of truth, "T".)

(2) There exist $x \leq y \in S$ and $z \in S$ such that $z \cdot x \nleq z \cdot y$ or $x \cdot z \nleq y \cdot z$. (This condition is called degree of falsity, "F".)

(3) There exist $x \leq y \in S$ and $z \in S$ such that $z \cdot x$ or $z \cdot y$ or $x \cdot z$ or $y \cdot z$ are indeterminate, or the relation between $z \cdot x$ and $z \cdot y$, or the relation between $x \cdot z$ and $y \cdot z$ are indeterminate. (This condition is called degree of indeterminacy, "I".)

Where (T, I, F) is different from $(1, 0, 0)$ that represents the classical Ordered Semigroup, and from $(0, 0, 1)$ that represents the AntiOrderedSemigroup.

Definition 3.4. Let $(S, \cdot, \leq)$ be a NeutroOrderedSemigroup . If "$\leq$" is a total order on A then A is called *NeutroTotalOrderedSemigroup*.

Definition 3.5. Let $(S, \cdot, \leq)$ be a NeutroOrderedSemigroup and $\emptyset \neq M \subseteq S$. Then

(1) M is a *NeutroOrderedSubSemigroup* of S if $(M, \cdot, \leq)$ is a NeutroOrderedSemigroup and there exist $x \in M$ with $(x] = \{y \in S : y \leq x\} \subseteq M$.

(2) M is a *NeutroOrderedLeftIdeal* of S if M is a NeutroOrderedSubSemigroup of S and there exists $x \in M$ such that $r \cdot x \in M$ for all $r \in S$.

(3) M is a *NeutroOrderedRightIdeal* of S if M is a NeutroOrderedSubSemigroup of S and there exists $x \in M$ such that $x \cdot r \in M$ for all $r \in S$

(4) M is a *NeutroOrderedIdeal* of S if M is a NeutroOrderedSubSemigroup of S and there exists $x \in M$ such that $r \cdot x \in M$ and $x \cdot r \in M$ for all $r \in S$.

(5) M is a *NeutroOrderedFilter* of S if $(M, \cdot, \leq)$ is a NeutroOrderedSemigroup and there exists $x \in S$ such that for all $y, z \in S$ with $x \cdot y \in M$ and $z \cdot x \in M$, we have $y, z \in M$ and there exists $y \in M$ $[y) = \{x \in S : y \leq x\} \subseteq M$.

Proposition 3.6. *Let $(S, \cdot, \leq)$ be a NeutroOrderedSemigroup and $\emptyset \neq M \subseteq S$. Then the following statements are true.*

(1) *If S contains a minimum element (i.e. there exists $m \in S$ such that $m \leq x$ for all $x \in S$.) and M is a NeutroOrderedSubSemigroup (or NeutroOrderedRightIdeal or NeutroOrderedLeftIdeal or NeutroOrderedIdeal) of S then the minimum element is in M.*

(2) *If If S contains a maximum element (i.e. there exists $n \in S$ such that $x \leq n$ for all $x \in S$.) and M is a NeutroOrderedFilter of S then M contains the maximum element of S.*

Proof. The proof is straightforward. □

Remark 3.7. Let $(S, \cdot, \leq)$ be a NeutroOrderedSemigroup. Then every NeutroOrderedIdeal of S is NeutroOrderedLeftIdeal of S and a NeutroOrderedRightIdeal of S. But the converse may not hold. (See Example 3.16.)

Definition 3.8. Let $(A, \star, \leq_A)$ and $(B, \circledast, \leq_B)$ be NeutroOrderedSemigroups and $\phi : A \to B$ be a function. Then

(1) ϕ is called *NeutroOrderedHomomorphism* if $\phi(x \star y) = \phi(x) \circledast \phi(y)$ for some $x, y \in A$ and there exist $a \leq_A b \in A$ with $a \neq b$ such that $\phi(a) \leq_B \phi(b)$.
(2) ϕ is called *NeutroOrderedIsomomorphism* if ϕ is a bijective NeutroOrderedHomomorphism.
(3) ϕ is called *NeutroOrderedStrongHomomorphism* if $\phi(x \star y) = \phi(x) \circledast \phi(y)$ for all $x, y \in A$ and $a \leq_A b \in A$ is equivalent to $\phi(a) \leq_B \phi(b) \in B$.
(4) ϕ is called *NeutroOrderedStrongIsomomorphism* if ϕ is a bijective NeutroOrdered-StrongHomomorphism.

Example 3.9. Let $S_1 = \{s, a, m\}$ and $(S_1, \cdot_1)$ be defined by the following table.

$\cdot_1$	s	a	m
s	s	m	s
a	m	a	m
m	m	m	m

Since $s \cdot_1 (s \cdot_1 s) = s = (s \cdot_1 s) \cdot_1 s$ and $s \cdot_1 (a \cdot_1 m) = s \neq m = (s \cdot_1 a) \cdot_1 m$, it follows that $(S_1, \cdot_1)$ is a NeutroSemigroup.
By defining the total order

$$\leq_1 = \{(m, m), (m, s), (m, a), (s, s), (s, a), (a, a)\}$$

on S_1, we get that $(S_1, \cdot_1, \leq_1)$ is a NeutroTotalOrderedSemigroup. This is easily seen as: $m \leq_1 s$ implies that $m \cdot_1 x \leq_1 s \cdot_1 x$ and $x \cdot_1 m \leq_1 x \cdot_1 s$ for all $x \in S_1$. And having $s \leq_1 a$ but $s \cdot_1 s = s \not\leq_1 m = a \cdot_1 s$.

Example 3.10. Let $S_2 = \{0, 1, 2, 3\}$ and $(S_2, \cdot_2)$ be defined by the following table.

$\cdot_2$	0	1	2	3
0	0	0	0	3
1	0	1	1	3
2	0	3	2	2
3	3	3	3	3

Since $0 \cdot_2 (0 \cdot_2 0) = 0 = (0 \cdot_2 0) \cdot_2 0$ and $1 \cdot_2 (2 \cdot_2 3) = 1 \neq 3 = (1 \cdot_2 2) \cdot_2 3$, it follows that $(S_2, \cdot_2)$ is a NeutroSemigroup.
By defining the total order

$$\leq_2 = \{(0,0),(0,1),(0,2),(0,3),(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$$

on S_2, we get that $(S_2, \cdot_2, \leq_2)$ is a NeutroTotalOrderedSemigroup. This is easily seen as: $0 \leq_2 3$ implies that $0 \cdot_2 x \leq_2 3 \cdot_2 x$ and $x \cdot_2 0 \leq_2 x \cdot_2 3$ for all $x \in S_2$. And having $1 \leq_2 2$ but $2 \cdot_2 1 = 3 \not\leq_2 2 = 2 \cdot_2 2$.

We present examples on NeutroOrderedSemigroups that are not NeutroTotalOrderedSemigroups.

Example 3.11. Let $S_2 = \{0, 1, 2, 3\}$ and $(S_2, \cdot_2^\prime)$ be defined by the following table.

$\cdot_2^\prime$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	3	2
3	0	1	3	2

Since $0 \cdot_2^\prime (0 \cdot_2^\prime 0) = 0 = (0 \cdot_2^\prime 0) \cdot_2^\prime 0$ and $2 \cdot_2^\prime (2 \cdot_2^\prime 3) = 3 \neq 2 = (2 \cdot_2^\prime 2) \cdot_2^\prime 3$, it follows that $(S_2, \cdot_2^\prime)$ is a NeutroSemigroup.
By defining the partial order (which is not a total order)

$$\leq_2^\prime = \{(0,0),(0,1),(0,2),(1,1),(2,2),(3,3)\}$$

on S_2, we get that $(S_2, \cdot_2^\prime, \leq_2^\prime)$ is a NeutroOrderedSemigroup (that is not a NeutroTotalOrderedSemigroup). This is easily seen as:
$0 \leq_2^\prime 1$ implies that $0 \cdot_2^\prime x = x \cdot_2^\prime 0 = 0 \leq_2^\prime 1 = 1 \cdot_2^\prime x = x \cdot_2^\prime 1$. And having $0 \leq_2^\prime 2$ but $2 \cdot_2^\prime 0 = 0 \not\leq_2^\prime 3 = 2 \cdot_2^\prime 2$.

Example 3.12. Let $S_3 = \{0, 1, 2, 3, 4\}$ and $(S_3, \cdot_3)$ be defined by the following table.

$\cdot_3$	0	1	2	3	4
0	0	0	0	3	0
1	0	1	2	1	1
2	0	4	2	3	3
3	0	4	2	3	3
4	0	0	0	4	0

Since $0 \cdot_3 (0 \cdot_3 0) = 0 = (0 \cdot_3 0) \cdot_3 0$ and $1 \cdot_3 (2 \cdot_3 1) = 1 \neq 4 = (1 \cdot_3 2) \cdot_3 1$, it follows that $(S_3, \cdot_3)$ is a NeutroSemigroup.
By defining the partial order

$$\leq_3 = \{(0, 0), (0, 1), (0, 3), (0, 4), (1, 1), (1, 3), (1, 4), (2, 2), (3, 3), (3, 4), (4, 4)\}$$

on S_3, we get that $(S_3, \cdot_3, \leq_3)$ is a NeutroOrderedSemigroup that is not NeutroTotalOrderedSemigroup as "$\leq_3$" is not a total order on S_3. This is easily seen as:
$0 \leq_3 4$ implies that $0 \cdot_3 x \leq_3 4 \cdot_3 x$ and $x \cdot_3 0 \leq_3 x \cdot_3 4$ for all $x \in S_3$. And having $0 \leq_3 1$ but $0 \cdot_3 2 = 0 \nleq_3 2 = 1 \cdot_3 2$.

Example 3.13. Let $\mathbb{Z}$ be the set of integers and define "$\odot$" on $\mathbb{Z}$ as follows: $x \odot y = xy - 1$ for all $x, y \in \mathbb{Z}$. Since $0 \odot (1 \odot 0) = -1 = (0 \odot 1) \odot 0$ and $0 \odot (1 \odot 2) = -1 \neq -3 = (0 \odot 1) \odot 2$, it follows that $(\mathbb{Z}, \odot)$ is a NeutroSemigroup. We define the partial order "$\leq_{\mathbb{Z}}$" on $\mathbb{Z}$ as $-1 \leq_{\mathbb{Z}} x$ for all $x \in \mathbb{Z}$ and for $a, b \geq 0$, $a \leq_{\mathbb{Z}} b$ is equivalent to $a \leq b$ and for $a, b < 0$, $a \leq_{\mathbb{Z}} b$ is equivalent to $a \geq b$. In this way, we get $-1 \leq_{\mathbb{Z}} 0 \leq_{\mathbb{Z}} 1 \leq_{\mathbb{Z}} 2 \leq_{\mathbb{Z}} \ldots$ and $-1 \leq_{\mathbb{Z}} -2 \leq_{\mathbb{Z}} -3 \leq_{\mathbb{Z}} \ldots$. Having $0 \leq_{\mathbb{Z}} 1$ and $x \odot 0 = 0 \odot x = -1 \leq_{\mathbb{Z}} x - 1 = 1 \odot x = x \odot 1$ for all $x \in \mathbb{Z}$ and $-1 \leq_{\mathbb{Z}} 0$ but $(-1) \odot (-1) = 0 \nleq_{\mathbb{Z}} -1 = 0 \odot (-1)$ implies that $(\mathbb{Z}, \odot, \leq_{\mathbb{Z}})$ is a NeutroOrderedSemigroup with -1 as minimum element.

Example 3.14. Let "$\leq$" be the usual order on $\mathbb{Z}$ and "$\odot$" be the operation define on $\mathbb{Z}$ in Example 3.13. One can easily see that $(\mathbb{Z}, \odot, \leq)$ is not a NeutroTotalOrderedSemigroup as there exist no $x \leq y \in \mathbb{Z}$ (with $x \neq y$) such that $z \odot x \leq z \odot y$ for all $z \in \mathbb{Z}$. In this case and according to Definition 3.3, $(T, I, F) = (0, 0, 1)$.

Example 3.15. Let $\mathbb{Z}$ be the set of integers and define "$\otimes$" on $\mathbb{Z}$ as follows: $x \otimes y = xy + 1$ for all $x, y \in \mathbb{Z}$. Since $0 \otimes (1 \otimes 0) = 1 = (0 \otimes 1) \otimes 0$ and $0 \otimes (1 \otimes 2) = 1 \neq 3 = (0 \otimes 1) \otimes 2$, it follows that $(\mathbb{Z}, \otimes)$ is a NeutroSemigroup. We define the partial order "$\leq_{\otimes}$" on $\mathbb{Z}$ as $1 \leq_{\otimes} x$ for all $x \in \mathbb{Z}$ and for $a, b \geq 1$, $a \leq_{\otimes} b$ is equivalent to $a \leq b$ and for $a, b \leq 0$, $a \leq_{\otimes} b$ is equivalent to $a \geq b$. In this way, we get $1 \leq_{\otimes} 2 \leq_{\otimes} 3 \leq_{\otimes} 4 \leq_{\otimes} \ldots$ and $1 \leq_{\otimes} 0 \leq_{\otimes} -1 \leq_{\otimes} -2 \leq_{\otimes} \ldots$. Having $0 \leq_{\otimes} -1$ and $x \otimes 0 = 0 \otimes x = 1 \leq_{\otimes} -x + 1 = -1 \otimes x = x \otimes (-1)$ for all $x \in \mathbb{Z}$ and

$1 \leq_{\otimes} 0$ but $1 \otimes 1 = 2 \nleq_{\otimes} 1 = 0 \odot 1$ implies that $(\mathbb{Z}, \otimes, \leq_{\otimes})$ is a NeutroOrderedSemigroup with 1 as minimum element.

We present some examples on NeutroOrderedSubSemigroups, NeutroOrderedRightIdeals, NeutroOrderedLeftIdeals, NeutroOrderedIdeals, and NeutroOrderedFilters.

Example 3.16. Let $(S_3, \cdot_3, \leq_3)$ be the NeutroOrderedSemigroup presented in Example 3.12. Then $I = \{0, 1, 2\}$ is a NeutroSubSemigroup of S_3 as $(I, \cdot_3)$ is NeutroOperation (with no AntiAxiom as $0 \cdot_3 (0 \cdot_3 0) = (0 \cdot_3 0) \cdot_3 0)$ and $0 \leq_3 1 \in I$ but $2 \cdot_3 0 = 0 \leq_3 4 = 2 \cdot_3 1$ is indeterminate over I as $4 \notin I$. Moreover, $(0] = \{0\} \subseteq I$. Since $g \cdot_3 0 = 0 \in I$ for all $g \in S_3$, it follows that I is a NeutroOrderedLeftIdeal of S_3. Moreover, having $1 \cdot_3 g \in \{0, 1, 2\} \subseteq I$ implies that I is a NeutroOrderedRightIdeal of S_3. Since there is no $g \in S$ satisfying $g \cdot_3 i \in I$ and $i \cdot_3 g \in I$ for a particular $i \in I$, it follows that I is not a NeutroOrderedIdeal of S_3.

Remark 3.17. Unlike the case in Ordered Semigroups, the intersection of NeutroOrdered-Subsemigroups may not be a NeutroOrderedSubsemigroup. (See Example 3.18.)

Example 3.18. Let $(S_3, \cdot_3, \leq_3)$ be the NeutroOrderedSemigroup presented in Example 3.12. One can easily see that $J = \{0, 1, 3\}$ is a NeutroOrderedSubsemigroup of S_3. From Example 3.16, we know that $I = \{0, 1, 2\}$ is a NeutroOrderedSubsemigroup of S_3. Since $(\{0, 1\}, \cdot_3)$ is a semigroup and not a NeutroSemigroup, it follows that $(I \cap J, \cdot_3, \leq_3)$ is not a NeutroOrdered-SubSemigroup of S_3. Here, $I \cap J = \{0, 1\}$.

Example 3.19. Let $(\mathbb{Z}, \odot, \leq_{\mathbb{Z}})$ be the NeutroOrderedSemigroup presented in Example 3.13. Then $I = \{-1, 0, 1, -2, -3, -4, \ldots\}$ is a NeutroOrderedIdeal of $\mathbb{Z}$. This is clear as:

(1) $0 \odot (1 \odot 0) = -1 = (0 \odot 1) \odot 0$ and $0 \odot (-1 \odot -2) = -1 \neq 1 = (0 \odot -1) \odot -2$;
(2) $g \odot 0 = 0 \odot g = -1 \in I$ for all $g \in \mathbb{Z}$;
(3) $-1 \in I$ and $(-1] = \{-1\} \subseteq I$;
(4) $0 \leq_{\mathbb{Z}} 1 \in I$ implies that $0 \odot x = x \odot 0 = -1 \leq_{\mathbb{Z}} x - 1 = x \odot 1 = 1 \odot x$ for all $x \in I$ and $-1 \leq_{\mathbb{Z}} 0 \in I$ but $-1 \odot -1 = 0 \nleq_{\mathbb{Z}} -1 = 0 \odot -1$.

Example 3.20. Let $(\mathbb{Z}, \odot, \leq_{\mathbb{Z}})$ be the NeutroOrderedSemigroup presented in Example 3.13. Then $F = \{-1, 0, 1, 2, 3, 4, \ldots\}$ is a NeutroOrderedFilter of $\mathbb{Z}$. This is clear as:

(1) $0 \odot (1 \odot 0) = -1 = (0 \odot 1) \odot 0$ and $1 \odot (2 \odot 3) = 4 \neq 2 = (1 \odot 2) \odot 3$;
(2) $1 \in F$ and for all $x \in \mathbb{Z}$ such that $x - 1 = 1 \odot x = x \odot 1 \in F$, we have $x \in F$;
(3) $0 \in F$ and $[0) = \{0, 1, 2, 3, 4, \ldots\} \subseteq F$;
(4) $0 \leq_{\mathbb{Z}} 1 \in F$ and $0 \otimes (-1) = -1 \leq -2 = 1 \otimes (-1)$ is indeterminate in F.

Here, F is not a NeutroOrderedSubSemigroup of $\mathbb{Z}$ as there exist no $x \in F$ with $(x] \subseteq F$.

Example 3.21. Let $(S_2, \cdot_2, \leq_2)$ be the NeutroTotalOrderedSemigroup presented in Example 3.10. Then $F = \{1, 2, 3\}$ is a NeutroOrderedFilter of S_2. This is clear as:

(1) $2 \cdot_2 (2 \cdot_2 2) = (2 \cdot_2 2) \cdot_2 2$ and $1 \cdot_2 (2 \cdot_2 1) = 3 \neq 1 = (1 \cdot_2 2) \cdot_2 1$;

(2) $1 \cdot_2 x \in F$ and $z \cdot_2 1 \in F$ implies that $x, z \in F$;

(3) $3 \in F$ and $[3) = \{3\} \subseteq F$;

(4) $2 \leq_2 3 \in F$ implies that $2 \cdot_2 x \leq_2 3 \cdot_2 x$ and $x \cdot_2 2 \leq_2 x \cdot_2 3$ for all $x \in F$ and $1 \leq_2 2$ but $2 \cdot_2 1 = 3 \not\leq_2 2 = 2 \cdot_2 2$.

Lemma 3.22. *Let $(S, \cdot, \leq_S)$ and $(S', \star, \leq_{S'})$ be NeutroOrderedSemigroups and $\phi : S \to S'$ be a NeutroOrderedStrongIsomorphism. Then S is a NeutroTotalOrderedSemigroup if and only if S' is a NeutroTotalOrderedSemigroup.*

Proof. The proof is straightforward. □

Remark 3.23. Let $(S, \cdot, \leq_S)$ and $(S', \star, \leq_{S'})$ be NeutroOrderedSemigroups and $\phi : S \to S'$ be a NeutroOrderedIsomorphism. Then Lemma 3.22 may not hold. (See Example 3.24.)

Example 3.24. Let $(S_2, \cdot_2, \leq_2)$ be the NeutroTotalOrderedSemigroup presented in Example 3.10, $(S_2, \cdot_2', \leq_2')$ be the NeutroOrderedSemigroup presented in Example 3.11, and $\phi : (S_2, \cdot_2, \leq_2) \to (S_2, \cdot_2', \leq_2')$ be defined as $\phi(x) = x$ for all $x \in S_2$. One can easily see that ϕ is a NeutroOrderedIsomorphism that is not NeutroOrderedStrongIsomorphism as: $\phi(0 \cdot_2 0) = \phi(0) = 0 = \phi(0) \cdot_2' \phi(0)$, $0 \leq_2 1$ and $\phi(0) = 0 \leq_2' 1 = \phi(1)$, $1 \leq_2 3$ but $\phi(1) = 1 \not\leq_2' 3 = \phi(3)$. Having $(S_2, \cdot_2, \leq_2)$ a NeutroOrderedSemigroup that is not NeutroTotalOrderedSemigroup and $(S_2, \cdot_2', \leq_2')$ a NeutroTotalOrderedSemigroup illustrates our idea.

Lemma 3.25. *Let $(S, \cdot, \leq_S)$ and $(S', \star, \leq_{S'})$ be NeutroOrderedSemigroups and $\phi : S \to S'$ be a NeutroOrderedStrongIsomorphism. Then S contains a minimum (maximum) element if and only if S' contains a minimum (maximum) element.*

Proof. The proof is straightforward. □

Remark 3.26. In Lemma 3.25, if $\phi : S \to S'$ is a NeutroOrderedIsomorphism that is not a NeutroOrderedStrongIsomorphism then S' may contain a minimum (maximum) element and S does not contain. (See Example 3.27.)

Example 3.27. With reference to Example 3.24, $(S_2, \cdot_2, \leq_2)$ has 0 as its minimum element whereas $(S_2, \cdot_2', \leq_2')$ has no minimum element.

Lemma 3.28. *Let $(S, \cdot, \leq_S)$ and $(S', \star, \leq_{S'})$ be NeutroOrderedSemigroups and $\phi : S \to S'$ be a NeutroOrderedStrongIsomorphism. If $M \subseteq S$ is a NeutroOrderedSubsemigroup of S then $\phi(M)$ is a NeutroOrderedSubsemigroup of S'.*

Proof. First, we prove that $(\phi(M),\star)$ is a NeutroSemigroup. Since $(M,\cdot)$ is a NeutroSemigroup, it follows that $(M,\cdot)$ is either NeutroOperation or NeutroAssociative.

- Case $(M,\cdot)$ is NeutroOperation. There exist $x,y,a,b\in M$ such that $x\cdot y\in M$ and $a\cdot b\notin M$ or $x\cdot y$ is indeterminate. The latter implies that there exist $\phi(x),\phi(y),\phi(a),\phi(b)\in\phi(M)$ such that $\phi(x)\star\phi(y)=\phi(x\cdot y)\in\phi(M)$ and $\phi(a)\star\phi(b)=\phi(a\cdot b)\notin\phi(M)$ or $\phi(x)\star\phi(y)=\phi(x\cdot y)$ is indeterminate.
- Case $(M,\cdot)$ is NeutroAssociative. There exist $x,y,z,a,b,c\in M$ such that $(x\cdot y)\cdot z=x\cdot(y\cdot z)$ and $(a\cdot b)\cdot c\neq a\cdot(b\cdot c)$. The latter implies that there exist $\phi(x),\phi(y),\phi(z),\phi(a),\phi(b),\phi(c)\in\phi(M)$ such that $(\phi(x)\star\phi(y))\star\phi(z)=\phi(x)\star(\phi(y)\star\phi(z))$ and $(\phi(a)\star\phi(b))\star\phi(c)\neq\phi(a)\star(\phi(b)\star\phi(c))$ (as ϕ is one-to-one.).

Since M is a NeutroOrderedSubsemigroup of S, it follows that there exist $x\in M$ such that $(x]\subseteq M$. It is easy to see that $(\phi(x)]\subseteq\phi(M)$ as for all $t\in S'$, there exist $y\in S$ such that $t=\phi(y)$. For $\phi(y)\leq_{S'}\phi(x)$, we have $y\leq_S x$. The latter implies that $y\in M$ and hence, $t\in\phi(M)$.

Since M is a NeutroOrderedSubsemigroup of S, it follows that:

(T) There exist $x\leq_S y\in M$ (with $x\neq y$) such that $z\cdot x\leq_S z\cdot y$ and $x\cdot z\leq_S y\cdot z$ for all $z\in M$;

(F) There exist $a\leq_S b\in M$ and $c\in M$ with $a\cdot c\nleq_S b\cdot c$ (or $c\cdot a\nleq_S c\cdot b$);

(I) There exist $x\leq_S y\in M$ and $z\in M$ with: $z\cdot x$ (or $x\cdot z$ or $y\cdot z$ or $z\cdot y$) indeterminate or $z\cdot x\leq_S z\cdot y$ (or $x\cdot z\leq_S y\cdot z$) indeterminate in M.

Where $(T,I,F)\neq(1,0,0)$ and $(T,I,F)\neq(0,0,1)$. This implies that

(T) There exist $\phi(x)\leq_{S'}\phi(y)\in\phi(M)$ (with $\phi(x)\neq\phi(y)$ as $x\neq y$) such that $\phi(z)\star\phi(x)\leq_{S'}\phi(z)\star\phi(y)$ and $\phi(x)\star\phi(z)\leq_{S'}\phi(y)\star\phi(z)$ for all $\phi(z)\in\phi(M)$;

(F) There exist $\phi(a)\leq_{S'}\phi(b)\in\phi(M)$ and $\phi(c)\in\phi(M)$ with $\phi(a)\star\phi(c)\nleq_{S'}\phi(b)\star\phi(c)$ (or $\phi(c)\star\phi(a)\nleq_{S'}\phi(c)\star\phi(b)$);

(I) There exist $\phi(x)\leq_{S'}\phi(y)\in\phi(M)$ and $\phi(z)\in\phi(M)$ with: $\phi(z)\star\phi(x)$ (or $\phi(x)\star\phi(z)$ or $\phi(y)\star\phi(z)$ or $\phi(z)\star\phi(y)$) indeterminate or $\phi(z)\star\phi(x)\leq_{S'}\phi(z)\star\phi(y)$ (or $\phi(x)\star\phi(z)\leq_{S'}\phi(y)\star\phi(z)$) indeterminate in $\phi(M)$.

Where $(T,I,F)\neq(1,0,0)$ and $(T,I,F)\neq(0,0,1)$. Therefore, $\phi(M)$ is a NeutroOrderedSubsemigroup of S'. □

Lemma 3.29. *Let $(S,\cdot,\leq_S)$ and $(S',\star,\leq_{S'})$ be NeutroOrderedSemigroups and $\phi:S\to S'$ be a NeutroOrderedStrongIsomorphism. If $M\subseteq S$ is a NeutroOrderedLeftIdeal (NeutroOrderedRightIdeal) of S then $\phi(M)$ is a NeutroOrderedLeftIdeal (NeutroOrderedRightIdeal) of S'.*

Proof. We prove that if $M \subseteq S$ is a NeutroOrderedLeftIdeal of S then $\phi(M)$ is a NeutroOrderedLeftIdeal of T. For NeutroOrderedRightIdeal, it is done similarly. Using Lemma 3.28, it suffices to show that there exist $z \in \phi(M)$ such that for all $t \in S'$ $t \star z \in \phi(M)$. Since M is a NeutroOrderedLeftIdeal of S, it follows that there exist $m \in M$ such that $s \cdot m \in m$ for all $s \in S$. Having ϕ an onto function implies that for all $t \in S'$, there exist $s \in S$ with $t = \phi(s)$. By setting $z = \phi(m)$, we get that $t \star z = \phi(s) \star \phi(m) = \phi(s \cdot m) \in \phi(M)$. □

Lemma 3.30. *Let* $(S, \cdot, \leq_S)$ *and* $(S', \star, \leq_{S'})$ *be NeutroOrderedSemigroups and* $\phi : S \to S'$ *be a NeutroOrderedStrongIsomorphism. If* $M \subseteq S$ *is a NeutroOrderedIdeal of* S *then* $\phi(M)$ *is a NeutroOrderedIdeal of* S'*.*

Proof. The proof is similar to that of Lemma 3.29. □

Example 3.31. Let $(\mathbb{Z}, \odot, \leq_{\mathbb{Z}})$ and $(\mathbb{Z}, \otimes, \leq_{\otimes})$ be the NeutroOrderedSemigroups presented in Example 3.13 and Example 3.15 respectively, and $\phi : (\mathbb{Z}, \odot, \leq_{\mathbb{Z}}) \to (\mathbb{Z}, \otimes, \leq_{\otimes})$ be defined as $\phi(x) = x+2$ for all $x \in \mathbb{Z}$. One can easily see that ϕ is a NeutroOrderedStrongIsomorphism. By Example 3.19, we have $I = \{-1, 0, 1, -2, -3, -4, \ldots\}$ is a NeutroOrderedIdeal of $(\mathbb{Z}, \odot, \leq_{\mathbb{Z}})$. Applying Lemma 3.30, we get that $\phi(I) = \{1, 2, 3, 0, -1, -2, \ldots\}$ is a NeutroOrderedIdeal of $(\mathbb{Z}, \otimes, \leq_{\otimes})$.

Lemma 3.32. *Let* $(S, \cdot, \leq_S)$ *and* $(S', \star, \leq_{S'})$ *be NeutroOrderedSemigroups and* $\phi : S \to S'$ *be a NeutroOrderedStrongIsomorphism. If* $M \subseteq S$ *is a NeutroOrderedFilter of* S *then* $\phi(M)$ *is a NeutroOrderedFilter of* S'*.*

Proof. Using Lemma 3.28, we get that $(\phi(M), \star)$ is a NeutroSemigroup and that $\leq_{S'}$ is NeutroOrder on $\phi(M)$. i.e., Conditions (1), (2), and (3) of Definition 3.3 are satisfied.
Since M is a NeutroOrderedFilter of S, it follows that there exist $x \in M$ such that $[x) \subseteq M$. It is easy to see that $[\phi(x)) \subseteq \phi(M)$ as for all $t \in S'$, there exist $y \in S$ such that $t = \phi(y)$. For $\phi(x) \leq_{S'} \phi(y)$, we have $x \leq_S y$. The latter implies that $y \in M$ and hence, $t \in \phi(M)$.
Since M is a NeutroOrderedFilter of S, it follows that there exist $x \in M$ such that for all $y, z \in S$ with $x \cdot y \in M$ and $z \cdot x \in M$ we have $y, z \in M$. The latter and having ϕ onto implies that there exist $t = \phi(x) \in \phi(M)$ such that for all $\phi(y), \phi(z) \in S'$ with $\phi(x) \star \phi(y) \in \phi(M)$ and $\phi(z) \star \phi(x) \in \phi(M)$ we have $\phi(y), \phi(z) \in \phi(M)$. □

Example 3.33. Let $(\mathbb{Z}, \odot, \leq_{\mathbb{Z}})$ and $(\mathbb{Z}, \otimes, \leq_{\otimes})$ be the NeutroOrderedSemigroups presented in Example 3.13 and Example 3.15 respectively, and $\phi : (\mathbb{Z}, \odot, \leq_{\mathbb{Z}}) \to (\mathbb{Z}, \otimes, \leq_{\otimes})$ be the NeutroOrderedStrongIsomorphism defined as $\phi(x) = x + 2$ for all $x \in \mathbb{Z}$. By Example 3.20, we have $F = \{-1, 0, 1, 2, 3, 4, \ldots\}$ is a NeutroOrderedFilter of $(\mathbb{Z}, \odot, \leq_{\mathbb{Z}})$. Applying Lemma 3.32, we get that $\phi(F) = \{1, 2, 3, 4, 5, 6, \ldots\}$ is a NeutroOrderedFilter of $(\mathbb{Z}, \otimes, \leq_{\otimes})$.

Lemma 3.34. *Let $(S, \cdot, \leq_S)$ and $(S', \star, \leq_{S'})$ be NeutroOrderedSemigroups and $\phi : S \to S'$ be a NeutroOrderedStrongIsomorphism. If $N \subseteq S'$ is a NeutroOrderedSubsemigroup of S' then $\phi^{-1}(N)$ is a NeutroOrderedSubsemigroup of S.*

Proof. Theorem 2.11 asserts that $\phi^{-1} : S' \to S$ is a NeutroOrderedStrongIsomorphism. Having $N \subseteq S'$ a NeutroOrderedSubsemigroup of S' and by using Lemma 3.28, we get that $\phi^{-1}(N)$ is a NeutroOrderedSubsemigroup of S. □

Lemma 3.35. *Let $(S, \cdot, \leq_S)$ and $(S', \star, \leq_{S'})$ be NeutroOrderedSemigroups and $\phi : S \to S'$ be a NeutroOrderedStrongIsomorphism. If $N \subseteq S'$ is a NeutroOrderedSubsemigroup of S' then $\phi^{-1}(N)$ is a NeutroOrderedLeftIdeal (NeutroOrderedRightIdeal) of S.*

Proof. Theorem 2.11 asserts that $\phi^{-1} : S' \to S$ is a NeutroOrderedStrongIsomorphism. Having $N \subseteq S'$ a NeutroOrderedLeftIdeal (NeutroOrderedRightIdeal) of S' and by using Lemma 3.29, we get that $\phi^{-1}(N)$ is a NeutroOrderedLeftIdeal (NeutroOrderedRightIdeal) of S. □

Lemma 3.36. *Let $(S, \cdot, \leq_S)$ and $(S', \star, \leq_{S'})$ be NeutroOrderedSemigroups and $\phi : S \to S'$ be a NeutroOrderedStrongIsomorphism. If $N \subseteq S'$ is a NeutroOrderedSubsemigroup of S' then $\phi^{-1}(N)$ is a NeutroOrderedIdeal of S.*

Proof. Theorem 2.11 asserts that $\phi^{-1} : S' \to S$ is a NeutroOrderedStrongIsomorphism. Having $N \subseteq S'$ a NeutroOrderedIdeal of S' and by using Lemma 3.35, we get that $\phi^{-1}(N)$ is a NeutroOrderedIdeal of S. □

Lemma 3.37. *Let $(S, \cdot, \leq_S)$ and $(S', \star, \leq_{S'})$ be NeutroOrderedSemigroups and $\phi : S \to S'$ be a NeutroOrderedStrongIsomorphism. If $N \subseteq S'$ is a NeutroOrderedFilter of S' then $\phi^{-1}(N)$ is a NeutroOrderedFilter of S.*

Proof. Theorem 2.11 asserts that $\phi^{-1} : S' \to S$ is a NeutroOrderedStrongIsomorphism. Having $N \subseteq S'$ a NeutroOrderedFilter of S' and by using Lemma 3.32, we get that $\phi^{-1}(N)$ is a NeutroOrderedFilter of S. □

We present our main theorems.

Theorem 3.38. *Let $(S, \cdot, \leq_S)$ and $(S', \star, \leq_{S'})$ be NeutroOrderedSemigroups and $\phi : S \to S'$ be a NeutroOrderedStrongIsomorphism. Then $M \subseteq S$ is a NeutroOrderedSubsemigroup of S if and only if $\phi(M)$ is a NeutroOrderedSubsemigroup of S'.*

Proof. The proof follows from Lemmas 3.28 and 3.34. □

Theorem 3.39. *Let $(S, \cdot, \leq_S)$ and $(S', \star, \leq_{S'})$ be NeutroOrderedSemigroups and $\phi : S \to S'$ be a NeutroOrderedStrongIsomorphism. Then $M \subseteq S$ is a NeutroOrderedLeftIdeal (NeutroOrderedRightIdeal) of S if and only if $\phi(M)$ is a NeutroOrderedLeftIdeal (NeutroOrderedRightIdeal) of S'.*

Proof. The proof follows from Lemmas 3.29 and 3.35. □

Theorem 3.40. *Let $(S, \cdot, \leq_S)$ and $(S', \star, \leq_{S'})$ be NeutroOrderedSemigroups and $\phi : S \to S'$ be a NeutroOrderedStrongIsomorphism. Then $M \subseteq S$ is a NeutroOrderedIdeal of S if and only if $\phi(M)$ is a NeutroOrderedIdeal of S'.*

Proof. The proof follows from Lemmas 3.30 and 3.36. □

Theorem 3.41. *Let $(S, \cdot, \leq_S)$ and $(S', \star, \leq_{S'})$ be NeutroOrderedSemigroups and $\phi : S \to S'$ be a NeutroOrderedStrongIsomorphism. Then $M \subseteq S$ is a NeutroOrderedFilter of S if and only if $\phi(M)$ is a NeutroOrderedFilter of S'.*

Proof. The proof follows from Lemmas 3.32 and 3.37. □

4. Conclusion

This paper contributed to the study of NeutroAlgebra by introducing, for the first time, NeutroOrderedAlgebra. The new defined notion was applied to semigroups and many interesting properties were proved as well illustrative examples were given on NeutroOrderedSemigroups.

For future research, it will be interesting to apply the concept of NeutroOrderedAlgebra to different algebraic structures such as groups, rings, modules, etc. and to study AntiOrderedAlgebra.

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University of New Mexico

Answer Note "A novel method for solving the fully neutrosophic linear programming problems: Suggested modifications"

Ranjan Kumar[1,*], Seyyed Ahmad Edalatpanah[2], Sudipta Gayen[3], and Said Broumi[4]

[1] Jain Deemed to be University, Jayanagar, Bengaluru, India; ranjank.nit52@gmail.com

[2] Department of Industrial Engineering, Ayandegan Institute of Higher Education, Iran

[3] National Institute of Technology Jamshedpur, India; sudi23dipta@gmail.com

[4] Faculty of Science Ben M'Sik, University Hassan II, Morocco

[*] Correspondence: ranjank.nit52@gmail.com

Abstract. Singh et al. [1], for solving the fully neutrosophic linear programming problems stated that the method of Abdel-Basset et al. [2] is scientifically incorrect and suggested a modified version for it. They have constructed their modifications based on different model assumptions which have been discussed in detail in the answer note. The purpose of this answer note is to inform that the justifications and clarifications given by Singh et al. [1] are not appropriate. We show that the outcome obtained by Abdel-Basset et al. [2] is correct and the method holds all conditions of the problem under neutrosophic environment. This answer note has also discussed some additional narrative which was not discussed in Abdel-Basset et al. [3] comment paper. Finally, we aim to bring back the faith in readers on the proposed method by Abdel-Basset et al. [2].

Keywords: Linear Programming Problem; Neutrosophic Linear Programming Problem; Note; reply

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1. Introduction

Fuzzy or neutrosophic or uncertain information are generally processed by transforming into an accurate number. This transformation can happen at the beginning of the decision process, or in the middle or final stage. Very recently, two articles [1,2] considered the solution of fully neutrosophic linear programming problems. However, the research by Singh et al. [1], and the research by Abdel-Basset et al. [2] are different due to the transformation of uncertain information into accurate numbers at different stages. Therefore, both the proposed methods hold good in their respective model and this implies that the modification suggested by Singh et al. [1] are not required being the conditions and provisions of these two models are itself different.

In this paper, the models considered by Abdel-Basset et al. [2] (Model 1, in Definition 1.1) and Singh et al. [1] (Model 2, in Definition 1.2) have been discussed in detail to understand the difference between the two proposed methods and therefore to justify our objective.

1.1. *Discussion on different models to handle LPP under neutrosophic environment*

Definition 1.1. (Section 3.1, Step 1, pp. 886-887) [1]: Abdel-Basset et al. [2] proposed Model 1 is as follows:

Maximize/Minimize $\left[\sum_{j=1}^{n} R\left(\tilde{C}_j\right) x_j\right]$

Subject to (P2)

Definition 1.2. (Section 4.1, Step 1, pp. 887): Singh et al. [1] consider the different model such as:

Maximize/Minimize $\left[R\left(\sum_{j=1}^{n} \tilde{C}_j x_j\right)\right]$

Subject to constraints.

where,

$\tilde{C}_j$= Neutrospohic cost value, and

x_{ij}= the neutrosophic variable.

R: rank function.

1.2. *Comparison of any two random neutrosophic numbers*

Definition 1.3. [2, 3]: Where $\hat{r}^N = \langle [\hat{r}_T, \hat{r}_I, \hat{r}_M, \hat{r}_E], (T_{\hat{r}}, I_{\hat{r}}, F_{\hat{r}})\rangle$ and $\hat{s}^N = \langle [\hat{s}_T, \hat{s}_I, \hat{s}_M, \hat{s}_E], (T_{\hat{s}}, I_{\hat{s}}, F_{\hat{s}})\rangle$ are two Trapezoidal neutrosophic numbers:

(a) $\hat{r}^N > \hat{s}^N$ iff $R\left(\hat{r}^N\right) > R\left(\hat{s}^N\right)$

(b) $\hat{r}^N < \hat{s}^N$ iff $R\left(\hat{r}^N\right) < R\left(\hat{s}^N\right)$

(c) $\hat{r}^N = \hat{s}^N$ iff $R\left(\hat{r}^N\right) = R\left(\hat{s}^N\right)$

Where $R\left(\hat{r}^N\right)$ is ranking function for $\hat{r}^N$ neutrosophic number.

1.3. *Discussion on arithmetic operation to handle LPP under neutrosophic environment*

To solve Model 1, Abdel-Baset et al. [2] use the following Definition 1.4.

Definition 1.4. [2]: Neutrosophic multiplication property with β ; where β is constant parameter;

$$\beta\tilde{\underline{m}} = \left\langle \left[\beta\tilde{m}^a, \beta\tilde{m}^s, \beta\tilde{m}^h, \beta\tilde{m}^o\right], \left(T_{\tilde{m}}, I_{\tilde{m}}, F_{\tilde{m}}\right)\right\rangle \text{ if } (\beta > 0)$$

Whereas, Singh et al. [1] use the below Definition 1.5 to solve Model 2.

Definition 1.5. [2]: Neutrosophic multiplication property with two neutrosophic numbers $\hat{r}^N$ and $\hat{s}^N$

$$\hat{r}^N \otimes \hat{s}^N = \begin{cases} \langle [\hat{r}_T \cdot \hat{s}_T, \hat{r}_I \cdot \hat{s}_I, \hat{r}_M \cdot \hat{s}_M, \hat{r}_E \cdot \hat{s}_E], (T_{\hat{r}} \wedge T_{\hat{s}}, I_{\hat{r}} \vee I_{\hat{s}}, F_{\hat{r}} \vee F_{\hat{s}})\rangle & \text{if } (\hat{r}_E > 0, \hat{s}_E > 0) \\ \langle [\hat{r}_T \cdot \hat{s}_E, \hat{r}_I \cdot \hat{s}_E, \hat{r}_M \cdot \hat{s}_I, \hat{r}_E \cdot \hat{s}_T], (T_{\hat{r}} \wedge T_{\hat{s}}, I_{\hat{r}} \vee I_{\hat{s}}, F_{\hat{r}} \vee F_{\hat{s}})\rangle & \text{if } (\hat{r}_E < 0, \hat{s}_E > 0) \\ \langle [\hat{r}_E \cdot \hat{s}_E, \hat{r}_M \cdot \hat{s}_M, \hat{r}_I \cdot \hat{s}_I, \hat{r}_T \cdot \hat{s}_T], (T_{\hat{r}} \wedge T_{\hat{s}}, I_{\hat{r}} \vee I_{\hat{s}}, F_{\hat{r}} \vee F_{\hat{s}})\rangle & \text{if } (\hat{r}_E < 0, \hat{s}_E < 0) \end{cases}$$

Where $\hat{r}^N = \langle [\hat{r}_T, \hat{r}_I, \hat{r}_M, \hat{r}_E], (T_{\hat{r}}, I_{\hat{r}}, F_{\hat{r}})\rangle$ and $\hat{s}^N = \langle [\hat{s}_T, \hat{s}_I, \hat{s}_M, \hat{s}_E], (T_{\hat{s}}, I_{\hat{s}}, F_{\hat{s}})\rangle$ are two Trapezoidal neutrosophic numbers.

2. Reply to suggested modification in the existing solution in Singh et al. [1]

When we consider the Model 1 (Definition 1.1) as suggested by the Abdel-Basset et al. [2], we need to follow the arithmetic property as mentioned in Definition 1.4. and when we consider the model 2 (Definition 1.2) as suggested by Singh et al. [1] we need to follow the Definition 1.5.

In this answer note, it is clear that Singh et al. [1], have suggested the modifications in Abdel-Basset et al. [2] which are not required. The authors have considered a different model (model 2) to modify the technique of Abdel-Basset et al. [2]. The conditions and provisions of these two methods are entirely different as the transformation of uncertainty information has taken place at different stages. So, based on above justification, we can say that Abdel-Basset et al. [2] method still holds all the required conditions necessary to solve all types of LPP problem under neutrosophic environment and do not needs any further modification.

2.1. *Second Irrelevant Condition of linearity of Singh et al. [1]*

Singh et al. [1] claims that the Abdel-Basset et.al. [2] proposed technique doesn't hold the linear property which is shown in Section 5, point i-ii, pp.888

We observe that Abdel-Basset et al. [2] already considered the above-mentioned demerit while drafting the manuscript. To justify our claim, we share a short observation below:

Observation: Abdel-Basset et al. [2] first found the score of each neutrosophic cost as shown in Definition 1.1, and then multiplied with neutrosophic number using the Definition 1.4. This helped in avoiding the above-discussed demerit of scoring property of neutrosophic set. Additionally, Singh et al. [1] has considered a different model (Definition 1.2) to justify their claim. The used model is wrong and nowhere used in the method proposed by Abdel-Basset et al. [2]. Hence, the model proposed by Singh et al. [1] doesn't prove any limitation of the technique proposed by Abdel-Basset et al. [2].

Conclusion

The method proposed by Abdel-Basset et al. [2] satisfy all the fundamental requirements for solving the problem under neutrosophic environment. Singh et al. [1] suggested some unnecessary changes ($P1 - P29$) with a different model for solving LPs where the transformation of uncertain information has taken place at different stage. So, the results mentioned in Abdel-Basset et al. [2] are still logical and legitimate. Hence, the model proposed by Singh et al. [1] doesn't prove any limitation of the technique proposed by Abdel-Basset et al. [2]. To motivate the readers, few relevant research articles [4–17] have been suggested to provide more insight on neutrosophic environment.

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University of New Mexico

Herbicides in the Irrigated Rice Production System in Babahoyo, Ecuador, Using Neutrosophic Statistics

Dalton Cadena-Piedrahita[1,2,*], Salomón Helfgott-Lerner[2], Andrés Drouet-Candel[3], Fernando Cobos-Mora[1], Nessar Rojas-Jorgge[1]

[1]Universidad Técnica de Babahoyo, Ecuador. Email: pcadena@utb.edu.ec
[2]Universidad Nacional Agraria la Molina, Escuela de Posgrado, Programa de Doctorado en Agricultura Sustentable, Perú
[3]Universidad Estatal Península de Santa Elena, Ecuador

Abstract. Rice cultivation is of great importance worldwide, due to its nutritional properties and because it is part of many plates of the inhabitants of all continents. Weeds are part of the factors that affect rice production, which is why it is necessary to apply herbicides, including the modality of the presence of herbicides in irrigation. This paper aims to carry out a statistical study on the effectiveness of different treatments to eliminate the weeds of the INDIA SFL 11 rice variety in Babahoyo, Ecuador. The evaluation of some results of the treatments was carried out visually based on both weed control and herbicide toxicity by using linguistic terms that are associated with an indeterminate scale of percentages, where the data are given in the form of intervals and not in crisp values. Additionally other seven criteria are also utilized. Thus, we decided to apply neutrosophic statistics as a study tool for this problem. Neutrosophic statistics extends classical statistics theory to the framework of neutrosophy, where intervals are used instead of analyzing crisp values. Specifically, Tukey's test is applied in comparing some data in form of intervals that denotes the imprecision of the obtained measurement.

Keywords: Weed, herbicides, rice, neutrosophic statistics, Tukey’s test.

1. Introduction

Rice (*Oryza sativa L*) is the most consumed grass in the world due to its high caloric content, which has led it to become the backbone of the economy of countries that depend directly on its production. According to the Food and Agriculture Organization (FAO), world production will be 514.9 millions of tons, surpassed by wheat with 757.4 millions of tons. ([1]).

Due to its characteristics, rice can be cultivated in different environments and areas. In Ecuador, it is carried out almost entirely on the coast, with 97% of the production, distributed mainly in three provinces: Guayas, Los Ríos and Manabí ([2]).

In Ecuador, rice cultivation is the largest, representing a third of the total area devoted to transitory products. During 2017, approximately 370,406 hectares were allocated ([3]).

Weeds in rice cause severe yield losses, affecting the number of tillers per plant, number of grains per panicle, and grain weight. In addition, they contribute to the survival of pests affecting the development of the crop and therefore increase production costs due to the necessary phytosanitary controls.

In Ecuador, the national average yield of rice in the first period 2018 was 4.81 t/ha, with the province of Loja being the one with the highest yield (9.10 t/ha), while in Los Ríos the lowest yield was obtained (3.64 t/ha). However, it represents an increase of 18% compared to the previous year.

According to the perspectives of the producers, there are many external factors that affect Ecuador's rice production, highlighting pests and/or diseases. 64% of farmers have been harmed by phytosanitary problems while 13% were affected by lack of water, weeds and salinity.

Due to the lack of knowledge, small farmers have used inappropriate techniques that are not optimal for the control of pests and diseases, allowing not only problems with their cultivation, but also to spreading of pests and diseases to other neighboring crops. Most rice producers do not coordinate planting, which results in having crops of different ages, facilitating the proliferation of pests. An incipient regulation and control of seeds, coming from abroad, by the State, facilitate the entry and proliferation of pests, [4].

Weeds are among the most limiting factors in rice production, as they cause direct and indirect damage to the crop through competition for light, water, and nutrients. They can reduce the quality of the harvest and are hosts for insects-pests and diseases. In addition, many weeds produce allelopathic compounds that are likely to affect normal crop growth.

The impact of weed damage is estimated to be between 15 and 20 percent of the production cost. The most important weeds in rice cultivation are grasses and within this group, *Echinochloa colona*, *Echinochloacrusgalli*, *Ischaemumrugosum* and *Leptochloaspp*. To this group of species must be added the non-commercial forms of *Oryza sativa* (black or red rice). The second group of weeds, in order of importance, are the sedges, among which *Cyperusesculentus*, *Cyperusferax*, *Cyperusiria* and *Fimbristilissp* stand out. These species are important since they are difficult to control and cause severe damage to the crop, [5].

There is a wide variety of herbicides used for weed control in rice cultivation. Among the recently introduced herbicides that inhibit protein synthesis through the inhibition of the enzyme acetolactate synthetase is halosulfuron-methyl, a sulfonylurea that is absorbed through the root system and the aerial part of the plant, and this is easily translocated within it, [5].

To use pre-emergent herbicides has been limited, because their effectiveness is conditioned by the moisture content and the preparation of the soil. However, with the early post-emergence application of residual herbicides, the advantage is that with a single application, emerged weeds can be controlled, while avoiding new weed emergencies. Therefore, it is considered that a good alternative for the control of rice weeds could be the application of herbicides in early post-emergence, [6].

We compare the results of applying eight herbicide treatments in the irrigated rice production system in Babahoyo. The evaluation of the results is carried out using the ALAM scale [2], where the evaluator associates a linguistic term with the effectiveness of the treatment. Each linguistic term is in turn associated with a percentage range of treatment effectiveness. Because visual inspection is indeterminate, intervals are taken as data rather than numerical values. This poses the challenge of conducting a statistical study based on intervals rather than crisps values. Also, the herbicide toxicity is evaluated using intervals. It is an opportunity to use neutrosophic statistics. The advantage of performing statistical processing using intervals is to obtaining greater accuracy at the cost of having greater indeterminacy.

Neutrosophic statistics is an extension of the classical statistics. While in classical statistics the data is known, formed by crisp numbers or parameters, in neutrosophic statistics the data has some indeterminacy[7-9]. In the neutrosophic statistics, the data may be ambiguous, vague, imprecise, incomplete, even unknown. Instead of crisp numbers used in classical statistics, one uses sets (that respectively approximate these crisp numbers) in neutrosophic statistics. Neutrosophic set theory has been used in agriculture and food problems, see [10-12].

This article is divided into the following sections. Section 2 is dedicated to recalling the main concepts of neutrosophic statistics and others necessaries concepts. Section 3 contains the study carried out in this research on the effectiveness of eight herbicide treatments on the INDIA SFL 11 rice variety, through the application of neutrosophic statistics and the Tukey's test, [13, 14]. Section 3 is dedicated to giving the conclusions of the paper.

2. Neutrosophy and Neutrosophic Statistics

This section is dedicated to describe some basic concepts of neutrosophy and neutrosophic statistics[9, 15-18].

Definition 1: ([19]) Let X be a universe of discourse. A *Neutrosophic Set* (NS) is characterized by three membership functions, $u_A(x), r_A(x), v_A(x) : X \to]^{-}0, 1^{+}[$, which satisfy the condition $^{-}0 \leq \inf u_A(x) + \inf r_A(x) + \inf v_A(x) \leq \sup u_A(x) + \sup r_A(x) + \sup v_A(x) \leq 3^{+}$ for all $x \in X$. $u_A(x), r_A(x)$ and $v_A(x)$ are the membership functions of truthfulness, indeterminacy and falseness of x in A, respectively, and their images are standard or non-standard subsets of $]^{-}0, 1^{+}[$.

Definition 2: ([19]) Let X be a universe of discourse. A *Single-Valued Neutrosophic Set* (SVNS) A on X is a set of the form:

$$A = \{\langle x, u_A(x), r_A(x), v_A(x) \rangle : x \in X\} \quad (1)$$

Where $u_A, r_A, v_A : X \to [0,1]$, satisfy the condition $0 \leq u_A(x) + r_A(x) + v_A(x) \leq 3$ for all $x \in X$. $u_A(x), r_A(x)$ and $v_A(x)$ denote the membership functions of truthfulness, indeterminate and falseness of x in A, respectively. For convenience a *Single-Valued Neutrosophic Number* (SVNN) will be expressed as $A = (a, b, c)$, where $a, b, c \in [0,1]$ and satisfy $0 \leq a + b + c \leq 3$.

Neutrosophic Statistics extends the classical statistics, such that we deal with set values rather than crisp values, [20].

Neutrosophic Descriptive Statistics is comprised of all techniques to summarize and describe the neutrosophic numerical data characteristics.

Neustrosophic Inferential Statistics consists of methods that permit the generalization from a neutrosophic sampling to a population from which it was selected the sample.

Neutrosophic Data is the data that contains some indeterminacy. Similarly to the classical statistics it can be classified as:

- *Discrete neutrosophic data*, if the values are isolated points.
- *Continuous neutrosophic data*, if the values form one or more intervals.

Another classification is the following:

- *Quantitative (numerical) neutrosophic data*; for example: a number in the interval (we do not know exactly), 47, 52, 67 or 69 (we do not know exactly);
- *Qualitative (categorical) neutrosophic data*; for example: blue or red (we don't know exactly), white, black or green or yellow (not knowing exactly).

The *univariate neutrosophic data* is a neutrosophic data that consists of observations on a neutrosophic single attribute.

Multivariable neutrosophic data is neutrosophic data that consists of observations on two or more attributes.

A *Neutrosophical Statistical Number* N has the form $N = d + i$, [21], where d is called *determinate part* and I is called *indeterminate part*.

A *Neutrosophic Frequency Distribution* is a table displaying the categories, frequencies, and relative frequencies with some indeterminacy. Most often, indeterminacies occur due to imprecise, incomplete or unknown data related to frequency. As a consequence, relative frequency becomes imprecise, incomplete, or unknown too.

Neutrosophic Survey Results are survey results that contain some indeterminacy.

A *Neutrosophic Population* is a population not well determined at the level of membership (i.e. not sure if some individuals belong or do not belong to the population).

A *simple random neutrosophic sample* of size n from a classical or neutrosophic population is a sample of n individuals such that at least one of them has some indeterminacy.

A *stratified random neutrosophic sampling* the pollster groups the (classical or neutrosophic) population by a strata according to a classification; afterwards the pollster takes a random sample (of appropriate size according to a criterion) from each group. If there is some indeterminacy, we deal with neutrosophic sampling.

Additionally we describe some concepts of interval calculus, which shall be useful in this paper.

Given $N_1 = a_1 + b_1 I$ and $N_2 = a_2 + b_2 I$ two neutrosophic numbers, some operations between them are defined as follows, [22, 23]:

$N_1 + N_2 = a_1 + a_1 + (b_1 + b_2)I$ (Addition),

$N_1 - N_2 = a_1 - a_1 + (b_1 - b_2)I$ (Difference),

$N_1 \times N_2 = a_1 a_2 + (a_1 b_2 + b_1 a_2 + b_1 b_2)I$ (Product),

$\frac{N_1}{N_2} = \frac{a_1+b_1 I}{a_2+b_2 I} = \frac{a_1}{a_2} + \frac{a_2 b_1 - a_1 b_2}{a_2(a_2+b_2)} I$ (Division).

A de-neutrosophication process gives an interval number $I = [a_1, a_2]$ for centrality, [24].

$$\lambda([a_1, a_2]) = \frac{a_1+a_2}{2} \qquad (2)$$

3. Results

This section is dedicated to provide the results of the present study. We use the Tukey's test, which is a type of ANOVA process, [13, 14, 25]. The inputs are N assessment divided in K groups each of them have the same number of elements. The null hypothesis test consists in asserting that the means of the groups are equal. It is sufficient that the mean of two groups are not significantly equal for rejecting the null hypothesis. The main Equation of the test is the following:

$$q_{observed} = \frac{M_{i.} - M_{j.}}{\sqrt{MS_{error}\left(\frac{1}{S}\right)}} \qquad (3)$$

Where $M_{i.}$ and $M_{j.}$ are the group means to compare, $MS_{error}$is the mean square error from the previously computed ANOVA test, and S is the number of observations per group.

$q_{observed}$ is compared with a $q_{critical}$ value calculated from a table of values (see [14]), which depends upon the α-level and the degrees of freedom $\nu = N - K$.

If $q_{observed} \leq q_{critical}$ we consider there is not a significant difference between ith and jth groups means, otherwise there exists a difference.

The present research was carried out in the town of San Pablo, Los Ríos province, located at 12 km of the Babahoyo-Montalvo road, with the geographical coordinates X = 672,825 Y = 979,717,5 and at 9 meters above sea level. Average annual precipitation is 2329.8 mm; 82 percent relative humidity; 998.2 hours of heliophany and temperature of 25.6 °C.

The place has a humid tropical climate, according to the Köppen climate classification with an average annual temperature of 25.5 °C. The average annual precipitation is 2,177.8 mm/year, relative humidity of 80.9 percent and 908.4 hours of heliophany on an annual average (UTB-2017 meteorological station).

The INDIA SFL 11 rice variety was used as sowing material, whose characteristics are presented in Table 1.

Plant height	126 cm
Tillering	Intermediate
Crop cycle	127–131 days
Yield potential	6 to 8 t/ha
Shelling	Intermediate
Weight of 1000 grains in shell	29 g
Pile rate	67 percent
Shelled grain size	7.52 mm
White center	None

Table 1: Characteristics of the INDIA SFL 11 rice variety.

Rice sowing was carried out on June 18, 2018 at a sowing density of 100 kg/ha. Eight treatments were evaluated, including one without any weed control (Table 2). The design used was that of complete random blocks with four repetitions and experimental units made up of 4x5 m plots, equivalent to 20 m^2 each of them, with a distance of 25 cm between rows and 25 cm between plants. For the evaluation and comparison of means of the treatments, the Tukey's test was used at 95 percent probability.

Pre-emergent	Dose/ha	Post-emergent	Dose / ha
T1. clomazone + butaclor	0.850 +1.4L	propanil + (picloram + 2.4-D amina)	2.3 + 0.4L
T2. clomazone + butaclor	0.850 +1.4L	manual weeding	3 weeds
T3. pendimentalin + bu-taclor	2.8 + 2.8L	cyhalofop	1.0L
T4. pendimentalin + bu-taclor	2.8 + 2.8L	manual weeding	3 weeds
T5.oxadiazon + butaclor	1.5 + 2.8L	(propanil + triclopyr)	5.0L
T6. oxadiazon + butaclor	1.5 + 2.8L	manual weeding	3 weeds
T7. clomazone + benti-ocarbo	0.850 + 4.0L	bispiribacsodium + (picloram + 2,4-D amina)	0.4 + 0.7L
T8. clomazone + benti-ocarbo	0.850 + 4.0L	manual weeding	3 weeds

Table 2: Treatments under study.

The application of the pre-emergency treatments was carried out on July 8th, 30 days after transplantation (dat). On both occasions, a 20-liter manual knapsack pump with a TEEJET 15004 flat fan nozzle was used, whose water consumption was 333 l/ha, after calibration. It was fertilized with urea (46% N), divided into equal parts 65.5 kg/ha at 15 dat and 65.5 kg/ha at 35 dat. Sulfur (ammonium sulfate 21% N and 24% S) was fractionated in equal parts, 10 kg/ha at 15 dat and 10 kg/ha at 35 dat. Phosphorus (DAP 18% N and 46% P_2O_5) in doses of 30 kg/ha and potassium 50 kg/ha (muriate of potassium 60% K_2O) were applied together in their entirety at 15 dat.

For preventive control of insects, thiamethoxam + lambdacyhalothrin was used, in doses of 250 cc/ha at 25 dat and then chlorpyrifos was used in doses of 750 cc/ha at 43 dat. In addition, for the preventive control of diseases, trifloxystrobin + tebuconazole was used, in doses of 600 cc/ha at 48 dat.

During the cultivation cycle the following variables were quantified:

1. Weed control. At 20 and 40 days after application (daa), the effect of the treatments on the weeds present was visually evaluated. For this, the scale proposed by ALAM ([2]) was used in which the value 0 indicates that the weeds were not affected and 100 that they were completely destroyed, Table 3.

Rating	Description
100 %	Full control
80-99 %	Excellent or very good
60-79 %	Good or sufficient

40-59 %	Doubtful or mediocre
20-39 %	Bad or lousy
0-19 %	Bad or null

Table 3: ALAM scale for weed control evaluation.

2. Toxicity to rice. The evaluation of the selectivity of the herbicide treatments to the crop was carried out on the same dates as the weed control evaluations, using the scale from 0 to 10, where 0 means that the rice did not suffer damage and 10 that all the plants are dead, see Table 4.

Rating	Description
0	No damage
1-3	Little damage
4-6	Moderate damage
7-9	Severe damage
10	Total Death

Table 4: Scale for herbicide toxicity evaluation.

3. Plant height. It was recorded at the moment of harvest, measuring in centimeters from the base of the plant to the apex of the most prominent panicle in 10 plants taken at random.

4. Number of tillers per m². It was randomly evaluated in 1.0 m² within the useful area of each experimental plot, counting the tillers at the time of harvest.

5. Panicles per m². For this variable, the number of panicles present in the same m² was determined that was used to count the tillers.

6. Panicle length. It was determined by measuring the distance between the ciliary node and the apex of the panicle, excluding the edges, in 10 random panicles.

7. Grains per panicle. In the same 10 panicles used in the previous variable, the number of grains in each panicle was counted.

8. Weight of 1000 grains. In the same 10 panicles used in the previous variable, the number of grains in each panicle was counted.

9. Crop yield. It was obtained based on the weight of the grains from the useful area of each experimental plot, standardizing at 14% humidity and transformed into kg/ha. To standardize the weights, the Azcon-Bieto formula was used, [26]:

$$Pu = Pa\frac{100 - ha}{100 - hd} \qquad (4)$$

Where:

Pu = uniform weight,

Pa = current weight,

ha = current humidity,

hd = desired humidity.

10. Economic Analysis. It was carried out based on the level of grain yield in kg/ha, with respect to the economic cost of the treatments.

Note that for the processing of the results that take values from Tables 3 and 4, the operations of the Tukey's test are generalized to the domain the interval values. The final results are converted into a single crisp value with the help of formula 2 using the de-neutrosophication process.

In Table 5, the averages of weed control at 20 and 40 daa of the products are recorded. In the analysis of variance, highly significant differences were obtained for both evaluations and the coefficients of variation were 2.88% and 5.38%, where the interval results were converted into crisp values by using Equation 2.

At 20 daa, the best weed control was reported in the mixture of clomazone + benthiocarb, in doses of 0.850L + 4.0L and bispyribacsodium + (picloram + 2,4-D amine) in doses of 0.4L + 0.7L with 95.8 percent weed control, being statistically superior to the other treatments.

At 40 daa, the mixture of clomazone + benthiocarb, in a dose of 0.850L + 4.0L and bispyribacsodium + (picloram + 2,4-D amine) in a dose of 0.4L + 0.7L presented the best weed control with 99 percent, being statistically equal to the mixtures of clomazone + butachlor in doses of 0.850L +1.4L and (propanil + picloram + 2,4-D amine) in doses of 2.3L + 0.4L.

Treatments	**20 daa**	**40 daa**
T1	[86.5, 88.5]b	[92.8, 94.8] ab
T2	[75.5, 77.5] d	[80.3, 82.3] c
T3	[82, 83] b	[85.7, 86.6] bc
T4	[79.6, 81.4] cd	[80.9, 81.7] c
T5	[86.79, 88.81] b	[89.5, 90.5] abc
T6	[77.95, 78.65] cd	[82.3, 82.7] c
T7	[95.3, 96.1] a	[98.5, 99.5] a
T8	[76.4, 76.6] d	[79.9, 80.1] c
General mean	[82.505, 83.820] (83.2)	[86.237, 87.275] (86.8)
Statistical significance	**	**
VC (%)	2.88	5.38

Means with the same letter do not differ significantly, according to the Tukey's Test.

Ns= not significant

*= significant

**= highly significant

Table 5: Percentage of control at 20 and 40 daa. The general means given between parenthesis is the de-neutrosophied value.

Herbicide selectivity

None of the treatments showed symptoms of phytotoxicity at 20 and 40 daa. That is to say, the specialists determined that no damage exists with respect to toxicity, based on the linguistic terms in Table 4.

Plant height

No differences were detected with a VC of 3.48% (Table 6).

With the mixture of clomazone + benthiocarb, in doses of 0.850L + 4.0L most weeds, the highest plant height (107.8 cm) was presented. The lowest height (99.5 cm) was obtained with the mixture of clomazone + benthiocarb in doses of 0.850L + 4.0L and with bispyribacsodium + (picloram + 2,4-D amine) in doses of 0.4L + 0.7L.

Number of tillers per m²

The means for this variable are observed in Table 6. The analysis of variance did not reach significant differences and its VC was 17.92%.

With the mixture of oxadiazon + butachlor in doses of 1.5L + 2.8L and (propanil + triclopyr) in doses of 5.0L, the highest number of 326.8 tillers/m² (326.8) was recorded. The lowest number (215.8) was obtained with the mixture of clomazone + benthiocarb, in doses of 0.850L + 4.0L and three manual weeds.

Panicles per m²

The means for this variable are presented in Table 6, not reaching significant differences and with a variation coefficient of 18.28%.

With the mixture of oxadiazon + butachlor in doses of 1.5L + 2.8L and (propanil + triclopyr) in doses of 5.0L, the highest number of panicles/m² (325.8) was achieved. The lowest number (210) was obtained with the mixture of clomazone + benthiocarb, in doses of 0.850L + 4.0L and three weeds.

Treatments	Plant height (cm)	Number of tillers/m²	Number of panicles/m²
T1	101.5	314.0	306.3
T2	105.9	302.5	290.8
T3	104.6	316.5	307.3
T4	107.4	301.8	290.5
T5	103.7	326.8	315.8
T6	104.7	255.8	241.5
T7	99.6	291.5	280.3
T8	107.8	215.8	210.0
General mean	104.4	290.6	280.3
Statistical significance	ns	ns	ns
Variation coefficient (%)	3.48	17.92	18.28

Table 6: Plant height, number of tillers, and panicles per m².

Crop yield

In Table 7 the averages for this variable are reported. The analysis of variance presented significant differences and the VC was 9.72%.

The mixture of oxadiazon + butachlor in doses of 1.5L+2.8L and (propanil + triclopyr) in doses of 5.0L reached the highest yield (4974.5 kg/ha), being statistically equal to the mixtures of clomazone + butachlor, in doses of 0.850 L +1.4L and propanil + (picloram + 2,4-D amine) in doses of 2.3L+ 0.4L; clomazone + butachlor, doses of 0.850L+1.4L and three weeds; pendimentalin + butacloren doses of 2.8L+2.8L and cyhalofopen doses of 1.0L; pendimentalin + butacloren doses of 2.8L+2.8L and three weeds; oxadiazon + butachloren doses of 1.5L +2.8L and three weeds; clomazone + benthiocarbo in doses of 0.850L+ 4.0L and bispyribacsodium + (picloram + 2,4-D amine) in doses of 0.4L+0.7L. All were statistically superior to the mixture of clomazone + benthiocarb in doses of 0.850L + 4.0L and three weeding, with a value of 3892.5 kg/ha.

Economic analysis

Table 7 presents the results of the economic analysis. The highest net benefit ($341.80) was registered with the mixture of clomazone + butachlor, in doses of 0.850L+1.4L and propanil + (picloram + 2,4-D amine), in doses of 2.3L+0.4L. The lowest value ($ -24.23) was obtained with the T8 clomazone + benthiocarb treatment, in doses of 0.850L+4.0L and three weeds.

Treatments	Yield (kg/ha)	Net benefit (USD)
T1	4917.5 ab	341.80
T2	4610.0 ab	237.20
T3	4944.8 ab	304.76
T4	4753.0 ab	254.50
T5	4974.5 a	181.71
T6	4532.5 ab	169.11
T7	4803.5 ab	236.49
T8	3892.5 b	-24.23
General mean	4678.5	
Statistical significance	*	
Variation coefficient (%)	9.72	

Table 7: Return and net profit.

Conclusion

This paper applied a statistical analysis to eight herbicides treatments in the irrigated rice production system in Babahoyo. We used the Tukey's test in a neutrosophic framework, because we preferred to maintain the imprecision obtained for indeterminate data for measuring the percentage of control at 20 and 40 days after application and the herbicide toxicity evaluation. We applied this test using intervals and later we de-neutrosiphied the final results. Thus we arrived to the following conclusions:

- The best weed control at 20 and 40 daa was obtained with the mixture of clomazone+benthiocarb, in doses of 0.850L+4.0L and bispyribacsodium + (picloram + 2.4 D amine) in doses of 0.4L+0.7L.

- Phytotoxicity was not observed with any of the treatments.
- Plant height and grains per panicle showed favorable results in the mixture of clomazone + benthiocarb in doses of 850L+4.0L and bispyribacsodium + (picloram + 2,4-D amine) in doses of 0.4L+ 0.7L.
- The mixture of oxadiazon + butachlor in doses of 1.5L+2.8L and (propanil + triclopyr) in doses of 5.0 L registered a greater number of tillers and panicles/m^2, panicle length, weight of 1000 grains and crop yield with 4974.5 kg/ha.
- The highest net benefit was registered with the mixture of clomazone + butachlor, in doses of 0.850L+1.4L and propanil + (picloram + 2,4-D amine), in doses of 2.3L+0.4L, with $341.80.

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